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INVARIANT SETS FOR NONLINEAR ELLIPTIC AND PARABOLIC SYSTEMS.(U)

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INVARIANT SETS FOR NONLINEAR ELLIPTIC AND PARABOLIC SYSTEMS

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6 INVARIANT SETS FOR NONLINEAR ELLIPTIC AND PARABOLIC SYSTEMS .

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ABSTRACT

In this paper we consider systems of weakly coupled nonlinear second order elliptic and parabolic equations with nonlinear, possibly coupled, boundary conditions. The aim is to find invariant sets of the form

$$S = \{(u_1,u_2,\ldots,u_m)\,|\,\varphi_i(x) \le u_i(x) \le \psi_i(x) \quad \text{a.e.}\}$$

for certain nonlinear reaction-diffusion equations:

$$U_t + LU = F(U) \quad \text{in } \Omega ,$$

$$BU = G(U) \quad \text{on } \partial\Omega ,$$

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where $L = (L_1,L_2,\ldots,L_m)$ (L_i a linear second order elliptic operator) and $B = (B_1,B_2,\ldots,B_m)$ (B_i a linear boundary operator of a general type) and $U = (u_1,u_2,\ldots,u_m)$. The main result essentially says that $S = \{U\,|\,\Phi \le U \le \Psi\}$ is an invariant set if

$$L\Phi \le F(\Phi) \quad \text{and} \quad L\Psi \ge F(\Psi) \quad \text{in } \Omega$$

and

$$B\Phi \le G(\Phi) \quad \text{and} \quad B\Psi \ge G(\Psi) \quad \text{on } \partial\Omega .$$

The work also includes some existence results for the parabolic problem and the associated nonlinear elliptic problem.

AMS (MOS) Subject Classifications - 35K55, 35K60, 35J55, 35J60

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SIGNIFICANCE AND EXPLANATION

Systems of coupled second order parabolic equations, or reaction-diffusion equations, arise in the mathematical models of various physical, chemical and biological processes. They describe the evolution in time of two or more substances which interact and diffuse. More specifically, reaction-diffusion equations arise in chemical reactor theory (where the components are concentrations of chemicals), in ecology (densities of species), in the theory of combusion (densities of fuel and thermal energy), and in the theory of nerve impulse transmission (densities of chemicals and electric charges). At any instant of time the densities, which are functions of the space variables, describe the state of the system. An invariant set S is a collection of states with the property that once the system is in a state which is a member of S then, however else the state will evolve, it will at all later times still be a member of S. Symbolically, if $U(t)$ is the state of time t then $U(t_0) \in S$
sub ∅ → an element of
implies $U(t) \in S$ for all $t > t_0$. In this paper criteria are found
an element of sub ∅
which can be used to find invariant sets which are described in terms of upper and lower bounds for the various components of the state. As a by-product existence and comparison results for the associated elliptic problems are obtained. In all results both the equations and the boundary conditions are allowed to be nonlinear.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

INVARIANT SETS FOR NONLINEAR ELLIPTIC AND PARABOLIC SYSTEMS

Hendrik J. Kuiper

1. Introduction.

Consider the reaction-diffusion equations

$$\text{(DE)} \qquad \frac{\partial u_k}{\partial t} + L_k u_k = f_k(x,t,U) \qquad 1 \le k \le m ,$$

where $L_1, L_2, \ldots, L_m$ are second order elliptic partial differential operators on a bounded open set $\Omega \subset R^n$ together with the conditions

$$\text{(BC)} \qquad B_k u_k = g_k(x,U) \qquad 1 \le k \le m ,$$

imposed on $U(t) = (u_1(\cdot,t), u_2(\cdot,t), \ldots, u_m(\cdot,t))$ at the boundary. For $t \in [0,T)$ we may think of $U(t)$ as belonging to some Banach space X of real valued functions from Ω into R^m. Let $K \subset [0,T) \times X$ be a set whose sections $K(t)$ are closed convex sets in X. K is then called an invariant set for the problem (DE)-(BC) if $U(t_0) \in K(t_0)$ implies that the solution $U(t) \in K(t)$ for all $t \in (t_0,T)$. Reaction-diffusion equations have lately received a great deal of attention. Their interest lies partially in the fact that they occur in the mathematical models for a wide range of natural processes (see e.g. [4], [5], [6], [24] and the references given in those papers). In particular there has been interest in the existence of invariant sets. Usually some restrictions are put on the form of K. For example, Weinberger [23] considered the case where $K(t)$ is independent of t and consists of functions which take on their values in some closed convex subset $C \subset R^m$. Unless the elliptic operators L_i are the same for all i more restrictions have to be placed on C ([1], [4]) such as requiring that $C = [\alpha_1,\beta_1] \times [\alpha_2,\beta_2] \times \ldots \times [\alpha_m,\beta_m]$. The present author [13] obtained results for invariant sets of the form $K(t) = \{(u_1,u_2,\ldots,u_m) \,|\, \varphi_i(x,t) \le u_i(x,t) \le \psi_i(x,t) \ \forall\ x \in \Omega\}$. In this paper similar results are obtained for the case where nonlinear boundary conditions are allowed.

In order to handle the nonlinear boundary conditions we use the nonlinear semigroup theory of Crandall, Liggett and Pazy which seems to be particularly well suited.

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This approach reduces the problem to one of studying invariant sets for associated elliptic problems. The solutions of the elliptic problem which we consider will be distributional solutions in $(H^1(\Omega))^m$. The solution of the reaction-diffusion equations which we will look at will also be of a weak type. This then means that the results on invariant sets which we obtain will also be valid for solutions of stronger type such as classical solutions or solutions $U \in C^1((0,T),(L^2(\Omega))^m) \cap C^0([0,T),D)$ where $D \subset (H^1(\Omega))^m$ is the domain of $(L_1,L_2,\ldots,L_m)$.

Although the main theorems (9, 10, 11, 16) can be read with only the aid of a few well marked definitions (the hypotheses being explicitly stated), we still feel it might be helpful to the reader if we state a somewhat simplified version of the invariant set theorem for the parabolic problem and give one simple application.

Let $L_k (k = 1,2,\ldots,m)$ be uniformly strongly elliptic with coefficients in $C^1(\bar{\Omega})$. Let the functions f_i and g_i be of class $C^1(\bar{\Omega} \times R^1 \times R^m)$ and assume the boundary conditions are of the form

$$B_k u_k \equiv \beta_k \cdot \nabla u_k + \gamma_k u_k = g_k(x,u_1,\ldots,u_m) \quad \text{on} \quad \partial\Omega ,$$

where β_k is a nowhere vanishing C^1 vector field on $\partial\Omega$ (which is assumed to be of class C^2) and $0 \leq \gamma_k \in C(\bar{\Omega})$ or the boundary condition may be of the Dirichlet type:

$$B_k u_k \equiv u_k(x) = g_k(x,u_1,\ldots,u_m) \equiv \theta_k(x) \quad \text{on} \quad \partial\Omega .$$

Let φ_i and ψ_i $(1 \leq i \leq m)$ be $C^1(\bar{\Omega}) \cap C^2(\Omega)$ functions which satisfy, for all $u_i \in C^1(\bar{\Omega})$ with $\varphi_i \leq u_i \leq \psi_i$:

$$\text{(A)} \quad \begin{aligned} L_i\varphi_i &\leq f_i(x,t,u_1,u_2,\ldots,u_{i-1},\varphi_i,u_{i+1},\ldots,u_m) \quad \text{in} \quad \Omega , \\ L_i\psi_i &\geq f_i(x,t,u_1,u_2,\ldots,u_{i-1},\psi_i,u_{i+1},\ldots,u_m) \quad \text{in} \quad \Omega , \\ B_i\varphi_i &\leq g_i(x,t,u_1,u_2,\ldots,u_{i-1},\varphi_i,u_{i+1},\ldots,u_m) \quad \text{on} \quad \partial\Omega , \\ B_i\psi_i &\geq g_i(x,t,u_1,u_2,\ldots,u_{i-1},\psi_i,u_{i+1},\ldots,u_m) \quad \text{on} \quad \partial\Omega . \end{aligned}$$

Then $\{(u_1,u_2,\ldots,u_m) | \varphi_i(x) \leq u_i(x) \leq \psi_i(x) \ \forall\, x \in \Omega,\ 1 \leq i \leq m\}$ is an invariant set for the problem (DE)-(BC).

Application. Let us consider a system of equations which arise in the theory of combustion (cf. [4], [11]). For simplicity we restrict ourselves to the one dimensional case:

$$n_t - k_1 n_{xx} = -ne^{-E/RT} ,$$
$$T_t - k_2 T_{xx} = Qne^{-E/RT} ,$$

where T and n denote the temperature and concentration of the fuel and where E,R,Q,k_1 and k_2 are constants (the calculations below can however still be carried out if we for example allow k_1 and k_2 to depend on x). We assume the region of interest is $x \in [0,L]$ and let us assume fuel is fed in at the right end and heat is lost at the left end: we impose the boundary conditions

$$n_x(0) = 0 \qquad n_x(L) = g(n)$$
$$T_x(0) = \beta T^\gamma \qquad T_x(L) = 0$$

where $g(0) = 0$ and $g(z) \geq 0$ whenever $z \geq \alpha_0$, where $\alpha_0 > 0$ is some constant. Suppose that we are given some initial conditions and that $T_0 = \max T(x,0)$, $n_0 = \max n(x,0)$ and that $\alpha = \max(\alpha_0, n_0, k_2 \beta T_0^\gamma/QL)$. We then claim that the set

$$I = \{(n,T) \,|\, 0 \leq n \leq n_0,\ 0 \leq T \leq \psi\}$$

is invariant if we choose

$$\psi(x) = (\tau/\beta)^{1/\gamma} + \tau x - \tau x^2/2L$$

where $\tau = Q\alpha L/k$. Let us verify that ψ satisfies the right inequalities (see (A)), the inequalities for the other functions used in the definition of I being trivially satisfied.

$$-k_2\psi''(x) = k_2\tau/L = Q\alpha \geq Qn_0 \geq Qne^{-E/R\psi}$$
$$\partial\psi(0)/\partial\nu = -\psi'(0) = -\tau = -\beta\psi(0)^\gamma$$
$$\partial\psi(L)/\partial\nu = \psi'(L) = \tau - \tau \geq 0$$

Also we note that $\psi(x) \geq \psi(0) = (\tau/\beta)^{1/\gamma} \geq T_0$ and therefore we know that

$$(n(\cdot,t),T(\cdot,t)) \in I \quad \text{for all} \quad t \geq 0 .$$

2. The Linear Elliptic Problem.

Let L_k, $1 \le k \le m$, be linear second order uniformly elliptic operators with real coefficients acting on real valued functions of $x = (x_1, x_2, \ldots, x_n)$ in a bounded open set $\Omega \subset R^n$.

$$L_k u \equiv -D_i[a_k^{ij}(x)D_j u + d_k^i(x)u] + b_k^i(x)D_i u + c_k(x)u$$

where summation is, and subsequently will be, carried out over any index which occurs both as a subscript and as a superscript within the same term. Next let B_k, $1 \le k \le m$, be first order boundary operators, of transversal order 1, acting on real valued functions defined on some subset Δ_k of the boundary $\partial\Omega$. In this section we shall look at the weakly coupled linear system

$$(L_k + \lambda)u_k(x) - h_k^j(x)u_j(x) = f_k(x) \qquad (x \in \Omega) , \tag{1}$$

with boundary conditions

$$B_k u_k(x) - e_k^j(x)u_j(x) = g_k(x) \qquad (x \in \Delta_k) , \tag{2}$$

$$u_k(x) = \theta_k(x) \qquad (x \in \Gamma_k \equiv \partial\Omega \backslash \Delta_k) , \tag{3}$$

for all $1 \le k \le m$. We will look at this problem from a variational point of view and hence it will be necessary for us to write the operator B_k in the form

$$B_k u = \nu_i[a_k^{ij}(x)D_j u + d_k^i u] + \sigma_k(x)u + t_k^i(x)D_i u$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$ is the unit outward normal on $\partial\Omega$ and $t_k = (t_k^1, t_k^2, \ldots, t_k^n)$ is a tangential vector field on $\partial\Omega$: $\nu_i(x)t_k^i(x) \equiv 0$ on $\partial\Omega$ for all $1 \le k \le m$.

We will use $(\, , \,)_Y$ to denote the usual $L^2(Y)$ inner product. When we take a direct sum of m copies of $L^2(Y)$ we shall still use the same symbol for the inner product on this direct sum, i.e. if $F = (f_1, f_2, \ldots, f_m)$ and $G = (g_1, g_2, \ldots g_m)$ are members of $\bigoplus_{i=1}^{m} L^2(Y)$ then $(F,G)_Y = \sum_{i=1}^{m} (f_i, g_i)_Y$. The norm is denoted by $\| \, \|_{0,Y}$. If $Y = \Omega$ we shall delete the subscript Ω. Hence $\| \, \|_0$ denotes the $L^2(\Omega)$ norm, $(\, , \,)$ the $L^2(\Omega)$ inner product (or the $(L^2(\Omega))^m$ norm and inner product respectively). The norm on the Sobolev space $W^{m,p}(\Omega)$ (derivatives of order $\le m$ are in $L^p(\Omega)$) is denoted by $\| \, \|_{m,p}$. If $p = 2$ we also use $\| \, \|_m$ to denote the norm on $H^m(\Omega) \equiv W^{m,2}(\Omega)$.

Corresponding script letters will be used to denote m-fold direct sums of function spaces e.g. $\mathcal{H}^1(\Omega) = \bigoplus_{i=1}^{m} H^1(\Omega)$, $\mathcal{C}^1(\bar{\Omega}) = \bigoplus_{i=1}^{m} C^1(\bar{\Omega})$ etc.

A formal integration by parts of the expression

$$\int_\Omega \sum_{k=1}^{\infty} [(L_k + \lambda)u_k]v_k dx ,$$

with u_k's which satisfy (1)-(3) and v_k's which vanish on Γ_k, leads to the equation

$$A_\lambda(U,V) \equiv \sum_{k=1}^{m} \{(a_k^{ij}D_j u_k, D_i v_k) + (d_k^i u_k, D_i v_k)$$

$$+ (b_k^i D_i u_k, v_k) + ((c_k + \lambda)u_k, v_k) - (h_k^j u_j, v_k)$$

$$+ (\sigma_k u_k, v_k)_{\Delta_k} - (e_k^j u_j, v_k)_{\Delta_k} + (t_k^i D_i u_k, v_k)_{\Delta_k}\}$$

$$= \sum_{k=1}^{m} \{(f_k, v_k) + (g_k, v_k)_{\Delta_k}\} ,$$

where $U = (u_1, u_2, \ldots, u_m)$ and $V = (v_1, v_2, \ldots, v_m)$. If the coefficients of L_k and B_k are sufficiently well behaved then the bilinear form A_λ is certainly well defined on $\mathcal{C}^1(\bar{\Omega}) \times \mathcal{C}^1(\bar{\Omega})$. We will impose conditions which will allow A_λ to be extended to a continuous $\mathcal{U}$-coercive form on $\mathcal{U} \times \mathcal{U}$ for some subspace $\mathcal{U}$ of $\mathcal{H}^1(\Omega)$.

(I): Ω is a bounded open set in R^n whose boundary is of class C^2.

This condition can be weakened to requiring that $\partial\Omega$ be Lipschitz continuous in a sense defined for example by Nečas [20]. However it seems this would require us to really handle the tangential derivatives in B_k rather than, as we shall be able to, remove them from consideration by treating another but equivalent problem. The nature of the work involved is then such that one might as well consider very general boundary operators, namely those which map $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$ (see e.g. [2]).

Let $\mathcal{D}(\Omega)$ be the $C^\infty(\Omega)$ functions of compact support, and $\mathcal{D}(\Omega)'$ the Schwartz distributions. The non-negatively valued functions in $\mathcal{D}(\Omega)$, denoted by $\mathcal{D}_+(\Omega)$, form a cone in $\mathcal{D}(\Omega)$. Let $\mathcal{D}_+(\Omega)'$ be the dual cone: $f \in \mathcal{D}_+(\Omega)'$ iff $f(\phi) \geq 0$ for all $\phi \in \mathcal{D}_+(\Omega)$. Consequently we have a partial order $\geq$ on $\mathcal{D}(\Omega)'$: $f \geq g$ iff

$f - g \in D_+(\Omega)'$. This partial order extends the usual partial order on $L^1(\Omega)$ functions: $f \geq g$ iff $f(x) \geq g(x)$ a.e. Furthermore we can extend such partial orders to vectors and matrices by saying $F \geq G$ if the relationship is satisfied component-wise.

We use γ_0 to denote the 0^{th} order trace map, i.e. the extension of the map $u \to u|_{\partial\Omega}$ from $C^1(\bar{\Omega})$ into $C^1(\partial\Omega)$ to a continuous map from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\partial\Omega) \subset L^p(\partial\Omega)$ for $p > 1$ ([1], [16], [17] or [18]).

We shall, on occasion, refer to the various Sobolev-Kondrasov imbedding results. We mention the following [15, p. 43]:

$$W^{r,p}(\Omega) \subset L^s(\Omega) \quad \text{if} \quad \frac{1}{s} \geq \frac{1}{p} - \frac{r}{n}, \; pr < n, \; s \geq 1 \; .$$

$$W^{r,p}(\Omega) \subset C^\alpha(\bar{\Omega}) \quad \text{if} \quad pr > n, \; \alpha < 1, \; a \leq \frac{pr-n}{p} \; .$$

The second imbedding is a compact linear map, as will be the first imbedding provided the inequality is strict.

We also need

(II): $\quad a_k^{ij} \in L^\infty(\Omega), \quad d_k^i \in L^q(\Omega), \quad b_k^i \in L^q(\Omega), \quad c_k \in L^{q/2}(\Omega), \quad D_i d_k^i \in L^{q/2}(\Omega)$,

$\quad 0 \leq h_k^j \in L^{q/2}(\Omega), \quad 0 \leq e_k^j \in L^p(\partial\Omega), \quad 0 \leq \sigma_k \in L^p(\partial\Omega), \quad t_k^i \in W^{1,q}(\Omega)$,

with $\operatorname{supp}\gamma_0 t_k^i \subset \bar{\Delta}_k$, where $p > n - 1$, $p > 1$, $q > n$ and $q \geq 2$. Also $\nu_i d_k^i \geq 0$ on $\partial\Omega$*), and there exists a constant μ_1 such that $c_k - D_i d_k^i \geq -\mu_1$.

The above hypotheses are directly related to the Sobolev imbedding theorems. We also require the operators L_k to be uniformly elliptic:

(III): There exists a positive constant ν_0 such that for every $1 \leq k \leq m$ and all $\xi \in R^n$ we have

$$a_k^{ij}(x)\xi_i\xi_j \geq \nu_0 \sum_{i=1}^{m} \xi_i^2 \; .$$

This condition can be weakened in order to treat certain degenerate-elliptic problems by methods described in [19].

*) i.e. $\int_\Omega (d_k^i D_i\phi + \phi D_i d_k^i)dx \geq 0$ whenever $0 \leq \phi \in H^1(\Omega)$.

(IV): Δ_k is an open subset of $\partial\Omega$ such that the $(n-1)$-dimensional Lebesgue measure of its boundary in $\partial\Omega$ is zero.

Let $W_0^{k,p}(\Omega)$ (resp. $H_0^k(\Omega)$) be the subspace of $W^{k,p}(\Omega)$ (resp. $H^k(\Omega)$) obtained by taking the closure of $\mathcal{D}(\Omega)$. The dual space of $W_0^{k,p}(\Omega)$ may be represented by $W^{-k,p^*}(\Omega)$, the collection of all Schwartz distributions of the form $D_i\psi^i + \psi$ with ψ_i and ψ in $L^{p^*}(\Omega)$ where $1/p^* + 1/p = 1$.

Before we proceed it should be noted that the usual Green's formula

$$(v, D_k w) = -(D_k v, w) + (v_k \gamma_0 v, \gamma_0 w)_{\partial\Omega}$$

which holds for $v, w \in H^1(\Omega)$, should be interpreted in the appropriate sense when $n = 1$. Although the results in this paper apply as well to the one-dimensional case we shall not take the trouble here to point out the various obvious notational modifications which need to be made.

For $S \subset \partial G$ let $H_S^1(G)$ be the closure in $H^1(G)$ of

$$\{u \in H^1(G) \,|\, u(x) = 0 \text{ a.e. on an open neighborhood of } \partial G \backslash S\}$$

With this notation $H_\phi^1(G) = H_0^1(G)$. If ∂G is sufficiently regular it can be shown (e.g. [10]) that this space is also equal to $\{u \in H^1(\Omega) \,|\, \gamma_0 u \equiv 0\}$. We shall use $H_\Delta^1(\Omega)$ to denote $\bigoplus_{k=1}^{m} H_{\Delta_k}^1(\Omega)$.

Our first objective will be to simplify our problem somewhat. Consider the bilinear functional A_λ. Using the Sobolev inequalities one easily shows that the first 5 terms are continuous on $H^1(\Omega) \times H^1(\Omega)$ (see e.g. [15]). Using the fact that if $u \in H^1(\Omega)$ then $\gamma_0 u \in H^{1/2}(\partial\Omega) \subset L^{(2n-2)/n-2}(\partial\Omega)$ (see e.g. [16] for the imbedding theorem for fractional Sobolev spaces) we also easily verify that the next two terms in the expression for $A_\lambda(U,V)$ are continuous bilinear functionals on $H^1(\Omega) \times H^1(\Omega)$. The last term is also continuous. However we can use the following result of Fiorenza [9] to remove it from consideration.

Theorem 1. Suppose Ω is a bounded open set in R^n, $n \geq 3$, whose boundary is of class C^2, and suppose $t_k^i \in W^{1,q}(\Omega)$. Then there exist functions

$\alpha_k^{ij}(=-\alpha_k^{ji}) \in L^\infty(\Omega)$ and functions $\gamma_k^i \in L^q(\Omega)$ such that for all $u,v \in H^1(\Omega)$ we have

$$(t_k^i D_i u, v)_{\partial\Omega} = (\alpha_k^{ij} D_j u, D_i v) + (\gamma_k^j D_j u, v)$$

Although in the proof Fiorenza assumes $t_k^i \in L^\infty(\Omega) \cap W^{1,n}(\Omega)$ $(\supset W^{1,q}(\Omega))$ and hence gets the γ_k^j's in $L^n(\Omega)$, an examination of the proof easily reveals that assuming our slightly more restrictive condition $t_k^i \in W^{1,q}(\Omega)$ does yield $\gamma_k^j \in L^q(\Omega)$. The proof for the case $n = 2$ is especially easy: Let s denote the distance along $\partial\Omega$, measured in such a way that when moving along the boundary in the direction of increasing s, Ω lies to the left of $\partial\Omega$. Let $\vec{t}$ be the unit tangent vector in the direction of increasing s, and let $\beta_k = \vec{t}_k \cdot \vec{t}$ $(\vec{t}_k = (t_k^1, t_k^2))$. The last term in the expression for $A_\lambda(U,V)$ takes the form

$$\sum_{k=1}^m \int_{\partial\Omega} \beta_k v_k (\nabla u_k \cdot \vec{t})\,ds = \sum_{k=1}^m \iint_\Omega \nabla(\beta_k v_k) \times \nabla u_k\,dxdy$$

$$= \sum_{k=1}^m \iint_\Omega [\beta_k(\nabla v_k \times \nabla u_k) + v_k(\nabla\beta_k \times \nabla u_k)]\,dxdy$$

$$= \sum_{k=1}^m \iint_\Omega [\alpha_k^{ij}(D_i u_k)(D_j v_k) + v_k \gamma_k^i D_i u_k]\,dxdy$$

Of course this proof requires that we extend $\vec{t}$ to a $W^{1,q}(\Omega)$ vector field. (Note that the product of two members of $W^{1,q}(\Omega)$ is again in $W^{1,q}(\Omega)$). We know however that $t_k^i \in C^1(\partial\Omega) \subset W^{1-1/q,q}(\partial\Omega)$ and hence the extension is possible by the trace theorem [16].

Using this theorem we can remove the last term in the expression (4) for $A_\lambda(U,V)$ and replace a_k^{ij} by $a_k^{ij} + \alpha_k^{ij}$ and b_k^i by $b_k^i + \gamma_k^i$. These new coefficients satisfy exactly the same hypotheses as the unaltered ones. Even the ellipticity constant ν_0 is preserved. Without loss of generality we shall from now on assume $t_k^i \equiv 0$ for all k and i.

<u>Lemma 2</u>. For any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for all $u \in H^1(\Omega)$

(i) $$\int_{\partial\Omega} u^2 dS \le \varepsilon \int_\Omega |Du|^2 dx + C(\varepsilon) \int_\Omega u^2 dx\,.$$

Moreover there exist constants $\varepsilon_0 > 0$ and $\mu_0 > 0$, independent of λ, such that

(ii) $$A_\lambda(U,U) \geq \varepsilon_0 \|U\|_1^2 + (\lambda - \mu_0)\|U\|_0^2 ,$$

i.e. A_λ is $H_\Delta^1(\Omega)$-coercive

Proof: We shall use the following result due to Lions [20]: If $X_a \subset X_b \subset X_c$ are Banach spaces with norms $\|\ \|_a, \|\ \|_b, \|\ \|_c$ respectively and if the first inclusion is compact linear and the second continuous linear, then for each $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\|x\|_b \leq \varepsilon \|x\|_a + C(\varepsilon)\|x\|_c \qquad \forall x \in X_a .$$

We now close $H^1(\Omega) \subset L^2(\Omega)$ with respect to the norm

$$\|u\|^2 = \int_{\partial\Omega} u^2 dS + \int_\Omega u^2 dx$$

and call this space H. Now we merely apply Lions' result to $H^1(\Omega) \subset H \subset L^2(\Omega)$. Of course this proof can also be accomplished by the standard partition of unity argument. For the proof of (ii) we note that the first term of $A_\lambda(U,U)$ satisfies

$$\sum_{k=1}^m \int_\Omega a_k^{ij}(D_j u_k)(D_i u_k)\,dx \geq \nu_0 \sum_{k=1}^m \sum_{i=1}^m \|D_i u_k\|_0^2 .$$

Hence it suffices to show that each of the other terms is dominated, in absolute value, by a quantity of the form

$$\varepsilon\|U\|_1^2 + C(\varepsilon)\|U\|_0^2$$

where the $\varepsilon > 0$ can be chosen arbitrarily small. This is easily seen to be the case. For example

$$\begin{aligned} |(d_k^i u_k, D_i u_k)| &\leq \|\varepsilon D_i u_k\|_0^2 + \|\tfrac{1}{\varepsilon} d_k^i u_k\|_0^2 \\ &\leq \varepsilon^2 \|\nabla U\|_0^2 + \varepsilon^{-2}\|d_k^i\|_{0,q}^2 \|u_k\|_{0,2q/(q-2)}^2 . \end{aligned}$$

By the Sovolev-Kondrasov embedding theorem the embedding $H^1(\Omega) \subset L^{2q/(q-2)}(\Omega)$ is compact continuous. Hence we can again use Lions' result to deduce that the above quantity is

$$\leq \varepsilon^2\|\nabla U\|_0^2 + \varepsilon^{-2}\|d_k^i\|_{0,q}\{\varepsilon^4\|\nabla u_k\|^2 + \tilde{C}(\varepsilon)\|u_k\|^2\} < \varepsilon\|\nabla U\|_0^2 + C(\varepsilon)\|U\|_0^2$$

provided ε is chosen sufficiently small. As another example let us take one of the integrals over Δ_k:

$$\left|\int_{\Delta_k} e_k^j u_j u_k dS\right| \leq \|e_k^j\|_{0,p,\Delta_k} \|u_j\|_{0,r,\partial\Omega} \|u_k\|_{0,r,\partial\Omega}$$

where $p > n - 1$, $r = 2p/(p - 1)$. This in turn is

$$\leq \text{cst.} \|U\|^2_{0,r,\partial\Omega}$$

We again apply Lions' result to $H^1(\Omega) \subset X \subset L^2(\Omega)$ where X is the closure of $H^1(\Omega)$ with respect to the norm

$$\|U\| = \sum_{i=1}^{m} \|\gamma_0 u_i\|_{0,r,\partial\Omega} + \|U\|_0$$

Since $H^1(\Omega), \subset L^2(\Omega)$ is a compact embedding and since $H^1(\Omega) \to H^{1/2}(\partial\Omega) \subset L^r(\partial\Omega)$ is a composition of a continuous linear map γ_0 and a compact embedding (since $\frac{1}{r} > \frac{1}{2} - \frac{1/2}{n-1}$ we can apply the Sobolev-Kondrasov results) we may conclude that

$$\left|\int_{\Delta_k} c_k^j u_j u_k dS\right| \leq \text{cst}\|U\| \leq \varepsilon\|U\|_1 + C(\varepsilon)\|U\|_0 \qquad \blacksquare$$

At this point it will be convenient to introduce some abbreviated notation. If $U = (u_1, u_2, \ldots, u_m) \in H^1(\Omega)$ then $HU = (h_1^i u_i, h_2^i u_i, \ldots, h_m^i u_i)$, $EU = (e_1^i u_i, e_2^i u_i, \ldots, e_m^i u_i)$. We also set $F = (f_1, f_2, \ldots, f_m)$, $G = (g_1, g_2, \ldots, g_m)$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $\Delta = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_m$, $\Gamma = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_m$ and L and B will respectively denote the operators $(L_1, L_2, \ldots, L_m)$ and $(B_1, B_2, \ldots, B_m)$. With this notation (1)-(3) can be written as

$$(L + \lambda - H)U = F \quad \text{in } \Omega, \tag{4}$$

$$(B - E)U = G \quad \text{on } \Delta, \tag{5}$$

$$U = \Theta \quad \text{on } \Gamma. \tag{6}$$

Definition. For $F \in H^1(\Omega)'$ (the dual space of $H^1(\Omega)$) and $G \in H^{-1/2}(\partial\Omega)$ (the dual space of $H^{1/2}(\Omega) = \gamma_0 H^1(\Omega)$) and $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$ we will define U to be a generalized solution of (1)-(3) if $U - \Theta \in H^1_\Delta(\Omega)$ and $A_\lambda(U,V) = (F,V) + (G,V)_\Delta$ for all $V \in H^1_\Delta(\Omega)$ (or equivalently for all $V \in C^\infty_\Delta(\Omega) = \{(v_1, v_2, \ldots, v_m) \in C^\infty(\Omega) \mid v_i = 0$ on an open neighborhood of Γ_i, $1 \leq i \leq m\}$). Of course every classical solution is a generalized solution and, for sufficiently large λ, there exists at most one generalized solution.

We shall need the following theorem of Stampacchia

Theorem 3. Let A be a continuous bilinear functional on a real Hilbert space Y with inner product $\langle\ ,\ \rangle$ and let $U \subset Y$ be a closed convex subset. Suppose A is strongly coercive on $U - U$, i.e. there is a positive constant c such that $A(y,y) \geq c\langle y,y\rangle$ for all $y \in U - U$. Let

$$U_z = \{y \in Y | z + \varepsilon y \in U \text{ for some } \varepsilon > 0\}$$

Then for each $f \in Y$ there exists a unique element $z \in U$ such that

$$A(z,y) \geq \langle f,y\rangle \text{ for all } y \in U_z .$$

The proof of this theorem can be found in [22] for the case where A is strongly coercive on all of Y. However, an examination of the proof shows that strong coerciveness on $U - U$ suffices. The minor modifications needed in the proof were pointed out in [12].

We use K to denote the cone of non-negatively valued functions in $H^1(\Omega)$. Consistent with our earlier notation, K will denote the cartesian product of m copies of K.

We remark here that the following two lemmas, 4 and 5, are true even if we impose no regularity conditions on $\partial\Omega$ or Δ. These two lemmas correspond to similar results obtained by Stampacchia [22]. First we need another definition.

Definition. Let U be a subspace of $H^1(\Omega)$, then $U \in H^1(\Omega)$ is called a U-subsolution for (1)-(3) if $A_\lambda(U,V) \leq 0$ for all $V \in U \cap K$.

Lemma 4. If U_1 and U_2 are $H^1_\Delta(\Omega)$-subsolutions, $\lambda > \mu_0$, μ_0 as in Lemma 2, and $W = \max(U_1,U_2)$, the component-wise maximum, then W is also a $H^1_\Delta(\Omega)$-subsolution.

Before we prove this lemma we need to make several observations whose proofs can be found in [15, pp. 50-54]. If k is a constant then the function $(u \vee k)(x) \equiv \max(u(x),k)$ is a member of $H^1(\Omega)$ whenever $u \in H^1(\Omega)$. Also, if $u_n \to u$ in $H^1(\Omega)$ then $u_n \vee k \to u \vee k$ in $H^1(\Omega)$. Moreover the distributional derivatives of $u \vee k$ satisfy

$$D_i(u \vee k)(x) = \begin{cases} 0 & \text{if } u(x) \leq k \\ D_i u(x) & \text{if } u(x) > k\,. \end{cases}$$

But since $u \vee v = u + (v - u) \vee 0 \in H^1(\Omega)$ if $u, v \in H^1(\Omega)$, we see that

$$D_i(u \vee v)(x) = \begin{cases} D_i u(x) & \text{if } u(x) \geq v(x) \\ D_i v(x) & \text{if } u(x) < v(x)\,. \end{cases}$$

Of course everything is modulo sets of measure zero; in particular $D_i u = D_i v$ a.e. on the set where $u = v$. Also analogous results hold if we replace $u \vee v$ by $u \wedge v \equiv \min(u,v)$.

<u>Proof</u>: Let $\mathcal{U} = \{U \in H^1(\Omega) | U \leq W$ and $U - W \in H^1_\Delta(\Omega)$ where $\leq$ should be interpreted as component-wise a.e. Clearly $\mathcal{U} - \mathcal{U} \subset H^1_\Delta(\Omega)$. For each $\Psi \in \mathcal{U}$ we define

$$\mathcal{U}_\Psi = \{V \in H^1_\Delta(\Omega) | \Psi + \varepsilon V \in \mathcal{U} \text{ for some } \varepsilon > 0\}\,.$$

We have the inclusions $\mathcal{U}_\Psi \subset H^1_\Delta(\Omega)$ and $-K \cap H^1_\Delta(\Omega) \subset \mathcal{U}_\Psi$. Now let Ψ be the unique element in $\mathcal{U}$ such that $A_\lambda(\Psi, Z) \geq 0$ for all $Z \in \mathcal{U}_\Psi$. This means that Ψ must be an $H^1_\Delta(\Omega)$-subsolution. Let $\Phi = \max(U_1, \Psi)$. We note that there exists an element $V \in H^1_\Delta(\Omega)$ such that $\Psi = V + W$. There exists a sequence $\{V_n\} \subset H^1(\Omega)$ such that for each i and n there exists an open neighborhood $N_{n,i}$ of Γ_i such that the i^{th} component of V_n vanishes on $N_{n,i}$ and such that $V_n \to V$ in $H^1(\Omega)$. From the above remarks we see that $\max(V_n + W, U_1) - (V_n + W)$ converges to $\Phi - \Psi$ in $H^1(\Omega)$, but also the i^{th} component of $\max(V_n + W, U_1) - (V_n + W)$ vanishes on $N_{n,i}$. Therefore $\Phi - \Psi \in H^1_\Delta(\Omega)$ and we have $\Phi - \Psi \in \mathcal{U}_\Psi$, so that

$$A_\lambda(\Psi, \Phi - \Psi) \geq 0 \tag{4}$$

We also claim that

$$A_\lambda(\Phi, \Phi - \Psi) \leq A_\lambda(U_1, \Phi - \Psi) \tag{5}$$

To see this we write

$$A_\lambda(\Phi - U_1, \Phi - \Psi) = \sum_{k=1}^{m} \{-(h_k^j(\varphi_j - u_{1j}), \varphi_k - \psi_k) - (e_k^j(\varphi_j - u_{1j}), \varphi_k - \psi_k)_{\partial\Omega}\}$$

which is indeed ≤ 0 since $\varphi_j \geq u_{1j}$ and $\varphi_k \geq \psi_k$ while h_k^j and e_k^j are ≥ 0. Combining (4) and (5) we obtain

$$A_\lambda(\Phi - \Psi, \Phi - \Psi) \leq A_\lambda(U_1, \Phi - \Psi) \leq 0$$

since U_1 is an $H^1_\Delta(\Omega)$-subsolution and $\Phi - \Psi \in K \cap H^1_\Delta(\Omega)$. Because $\lambda > \mu_0$ we see that $\Phi = \Psi$ and hence $U_1 \leq \Psi$. Similarly it follows that $U_2 \leq \Psi$ and consequently $W = \Psi$, an $H^1_\Delta(\Omega)$-subsolution. ∎

Lemma 5. If U is a generalized solution of (1)-(3) with $\lambda > \mu_0$, $F \geq 0$, $G \geq 0$ and $\Theta \geq 0$, then $U \geq 0$.

Proof: Both $-U$ and 0 are $H^1_\Delta(\Omega)$-subsolutions and hence so is $W = \max(0,-U)$. But since W is also in $K \cap H^1_\Delta(\Omega)$ we see that $A_\lambda(W,W) \leq 0$ and therefore $W = 0$. ∎

Remark. If $f \in L^1(\Omega)$ and $f \geq 0$ a.e. then $f \in D_+(\Omega)$ but the converse is not generally true. However since $\Theta \in L^\infty(\Omega)$ one can easily show that $\theta_k \in D_+(\Omega)$ implies $\theta_k \geq 0$ a.e. by merely taking a sequence in $D_+(\Omega)$ which converges in $L^1(\Omega)$ to the characteristic function χ of the set $\{x | \theta_k(x) < 0\}$. Therefore $(\theta_k, \chi) = 0$.

Theorem 6. Problem (1)-(3) has a unique generalized solution for each $F \in H^1(\Omega)'$, $G \in H^{-1/2}(\Omega)$ and $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$ provided $\lambda \geq \mu_0$.

Proof: This is a simple application of Theorem 3. Let U be the affine space $\Theta + H^1_\Delta(\Omega)$ and Y the Hilbert space $H^1(\Omega)$. By the Riesz representation theorem there exists a $T \in H^1(\Omega)$ such that $\langle T,U\rangle = (F,U) + (G,U)_{\partial\Omega}$ for all $U \in H^1(\Omega)$ where $\langle\ ,\ \rangle$ is the usual inner product on $H^1(\Omega)$ extended in the obvious manner to the direct sum of such spaces. Hence, since $U - U = H^1_\Delta(\Omega)$, there exists a unique $U \in U$ such that

$$A_\lambda(U,V) \geq \langle T,V\rangle \quad \forall V \in H^1_\Delta(\Omega) .$$

But since $-V \in H^1_\Delta(\Omega)$ we have in fact equality.

Using the Sobolev embedding theorem one finds that $L^{q/2}(\Omega) \subset H^1(\Omega)'$ and $L^p(\partial\Omega) \subset (\gamma_0 H^1(\Omega))'$. This justifies the following definition. ∎

Definitions. Let H denote the space $H^1(\Omega) \cap L^\infty(\Omega)$ with norm $|||U||| = \|U\|_{1,2} + \|U\|_{0,\infty}$ and let G_λ be the map from $L^{q/2}(\Omega) \times L^p(\Delta) \times H$ into $H^1(\Omega)$ which associates with each triple (F,G,Θ) the unique solution U to (1)-(3) $(\lambda > \mu_0)$.

Theorem 7. Suppose $\lambda > \mu_0$ and U is a generalized solution to (1)-(3) with $F \in L^{q/2}(\Omega)$, $G \in L^p(\partial\Omega)$ and $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$ then $U \in L^\infty(\Omega)$.

Proof: In order to apply known results for single component equations we first consider the case where $H = 0$ and $E = 0$. It will suffice to show that the solution is bounded from above. Let $\mu > \|\Theta\|_{0,\infty}$ and $M = (\mu,\mu,\ldots,\mu)$, then $U' = U - M$ satisfies

$$(L + \lambda)U' = F_1 \in L^{q/2}(\Omega) \quad \text{in } \Omega\,,$$
$$BU' \le G_1 \in L^p(\partial\Omega) \quad \text{on } \Delta\,,$$
$$U' = \Theta - M \le 0 \quad \text{on } \Gamma\,,$$

where $F_1 = (f_{1i}, f_{12}, \ldots, f_{1m})$ with $f_{1i} = f_i + \mu D_j d_i^j - \mu c_i - \mu\lambda$ and $G_1 = (g_{1i}, g_{12}, \ldots, g_{1m})$ with $g_{1i} = g_i - \mu\sigma_i$. Applying lemma 5 we see that $U' \le V$ where

$$\begin{aligned}(L + \lambda)V &= F_2 \quad \text{in } \Omega\,,\\ BV &= G_2 \quad \text{on } \partial\Omega\,,\end{aligned} \tag{A}$$

where F_2 (resp. G_2) consists of the absolute values of the components of F_1 (resp. G_1). Let V_1 be the generalized solution of

$$\begin{aligned}(L + \lambda)V_1 &= F_2 \in L^{q/2}(\Omega)\\ \partial V_1/\partial N &= 0 \quad \text{on } \partial\Omega\end{aligned} \tag{B}$$

where $\partial/\partial N = \nu_i(a_k^{ij}D_j + d_k^i)$. We can apply a result of Stampacchia [21] which states that the solution u of

$$(L_k + \lambda)u = f \in W^{-1,\sigma}(\Omega)$$
$$\partial u/\partial N = 0 \quad \text{on } \partial\Omega$$

will be in $L^\rho(\Omega)$ if $\rho^{-1} > \sigma^{-1} - n^{-1}$ ($\rho = \infty$ is allowed, setting $1/\infty = 0$). By the Sobolev embedding theorem $W_0^{1,q^*} \subset L^{(q/2)^*}$ where $\alpha^* \overset{\text{def}}{=} \alpha(\alpha - 1)^{-1}$. Therefore $L^{q/2} \subset W^{-1,q}(\Omega)$ and, since $q^{-1} - n^{-1} < 0$ we have $V_1 \in L^\infty(\Omega)$. Recalling that $\sigma_k \ge 0$ and noting $V \ge 0$ we see that $V - V_1 \le V_2$ where

$$\begin{aligned}(L + \lambda)V_2 &= 0\\ \partial V_2/\partial N &= G_2 \in L^p(\partial\Omega)\end{aligned} \tag{C}$$

Another regularity result, due to Murthy and Stampacchia [19] tells us that $V_2 \in L^\infty(\Omega)$ since $p > n - 1$ (the work of Murthy and Stampacchia deals with a more complicated problem namely certain degenerate elliptic problems. Also the theorem we need [19, p. 61] contains some minor, but confusing, errors. For these reasons we have included a proof of this result in an appendix). The regularity results which we used were proven for single component problems. When one includes coupling terms $-HU$ and $-EU$ on the right hand side of (A) they are no longer a priori in $L^{q/2}(\Omega)$ and $L^{p}(\partial\Omega)$ respectively (unless $n \leq 3$). Although we will not take this route, we do note that one can treat (B), with coupling $-HU$, by bootstrapping, showing that if $U \in L^{\rho}(\Omega)$, $\rho < q/2$, then $U \in L^{\rho+\varepsilon}(\Omega)$ for some $0 < \varepsilon \leq q/2 - \rho$ etc. The proof of the regularity result for (C) is relatively simple and can be easily extended to the case where we introduce a coupling term $-EU$. However to use this approach to deal with the case where we have both coupling terms present is rather lengthy (unless $n \leq 3$). Therefore, we will use a different approach. Let $n - 1 < p' < p$ and $n < q' < q$ and r so large that $r^{-1} + p^{-1} < (p')^{-1}$ and $r^{-1} + (q/2)^{-1} < (q'/2)^{-1}$. Let G be the map from $L^{q'/2}(\Omega) \times L^{p'}(\partial\Omega)$ into H defined by

$$(L + \lambda)G(F,G) = F \in L^{q'/2}(\Omega) \quad \text{in } \Omega ,$$
$$BG(F,G) = G \in L^{p'}(\Omega) \quad \text{on } \partial\Omega .$$

One easily sees that G is a closed linear operator and hence continuous by the closed graph theorem. We claim the map $G \circ P \circ I$ defined by the diagram below is a compact continuous linear map from H into itself:

$$U \in H \xrightarrow{I} (U,\gamma_0 U) \in L^{r}(\Omega) \times L^{r}(\partial\Omega) \xrightarrow{P} (HU,EU) \in L^{q'/2}(\Omega) \times L^{p'}(\partial\Omega) \xrightarrow{G} V \in H .$$

To see this we note that by Hölder's inequality P is bounded linear while I is obviously continuous. But I is also compact for if $\{U_n\}$ is bounded there must exist a subsequence $\{U_{n_i}\}$ such that both U_{n_i} and $\gamma_0 U_{n_i}$ converge in $L^2(\Omega)$ and $L^2(\partial\Omega)$. If $U_{n_i} \to U$ then a fortiori $\gamma_0 U_{n_i} \to \gamma_0 U$. It may furthermore be assumed, without loss of generality that $U_{n_i} \to U$ a.e. in Ω and $\gamma_0 U_{n_i} \to \gamma_0 U$ a.e. in $\partial\Omega$. But we also have things bounded a.e. and therefore, applying the dominated convergence theorem, we have $U_{n_i} \to U$ in $L^{r}(\Omega)$ and $\gamma_0 U_{n_i} \to \gamma_0 U$ in $L^{r}(\partial\Omega)$. We next consider, in H, the equation

$$U - G \circ P \circ IU = G(F_1,G_1) . \tag{D}$$

According to the Fredholm theory this equation has a solution in H if Ker (id. + $G \circ P \circ I$) is trivial. But if U_0 were in the kernel then one easily sees that $(L + \lambda - H)U_0 = 0$ and $(B - E)U_0 = 0$ with $\lambda > \mu_C$. By lemma 5 $U_0 = 0$. Therefore (D) has a solution $U \in H$ which is also a solution of (1)-(3). By uniqueness we are done. ■

<u>Theorem 8.</u> $\underset{\sim}{G}_\lambda : H \to H$ is a continuous map which is <u>monotone</u> in the sense that $F_1 \leq F_2$ a.e., $G_1 \leq G_2$ a.e. and $\Theta_1 \leq \Theta_2$ a.e. implies $\underset{\sim}{G}_\lambda(F_1,G_1,\Theta_1) \leq \underset{\sim}{G}_\lambda(F_1,G_1,\Theta_1)$ a.e.

<u>Proof:</u> The monotonicity is of course a direct consequence of linearity and lemma 5. Let $U_i = G_\lambda(F_i,G_i,\Theta)$, $i = 1,2$, then $U_1 - U_2 \in H^1_\Delta(\Omega)$ and hence, letting $\varepsilon = \min(\varepsilon_0, \lambda - \mu_0)$,

$$\varepsilon\|U_1 - U_2\|_1^2 \leq A_\lambda(U_1 - U_2, U_1 - U_2)$$

$$(F_1 - F_2, U_1 - U_2)_\Omega + (G_1 - G_2, \gamma_0 U_1 - \gamma_0 U_2)_{\partial\Omega} \leq$$

$$\text{cst.}\{\|F_1 - F_2\|_{0,q/2} + \|G_1 - G_2\|_{0,p,\Delta}\}\|U_1 - U_2\|_1 ,$$

where we used the Sobolev and Hölder inequalities. Hence for fixed Θ, $\underset{\sim}{G}_\lambda$ must be a closed linear operator. Applying the closed graph theorem we have continuity with respect to (F,G). To conclude the proof it suffices to show that $\underset{\sim}{G}_\lambda$ is continuous with respect to Θ for $F \equiv 0$ and $G \equiv 0$. Again this reduces to showing that the graph $\{(\Theta, \underset{\sim}{G}_\lambda(0,0,\Theta)\}$ is closed. To see this suppose $\Theta_i \to 0$ and $\underset{\sim}{G}_\lambda(0,0,\Theta_i) \to W$ then, extending standard arguments (see e.g. [2]) to the multi-component case (cf. equation (4) it can be shown that

$$0 = \int_\Omega (L + \lambda - H)U_i W \, dx = A_\lambda(U_i,W) - \int_\Gamma (\partial U_i/\partial N) W \, dS$$

where $\partial/\partial N : H^1(\Omega) \to H^{-1/2}(\partial\Omega)$ is a continuous linear map corresponding to $\{\nu_i(a_k^{ij}D_j + d_k^i)\}_{k=1}^m$ (Since we will never use continuity with respect to Θ_i we omit the details). Therefore, for $\lambda > \mu_0$ we have

$$\varepsilon_0\|U_i\|_1^2 \leq A_\lambda(U_i,U_i) = \int_\Gamma (\partial U_i/\partial N)\gamma_0\Theta_i \, dS \leq \text{cst.} \times \|U_i\|_1\|\Theta_i\|_1$$

We might point out that one can show, using the methods used early in the proof of the previous theorem, that the graph is closed in the L^2 topology: $\|U_i\| \leq \text{cst.} \times \|\Theta\|_{2,1,\Omega}$. Either of these inequalities can be used, in conjunction with the closed graph theorem or these inequalities may be used together without resorting to the closed graph theorem. ∎

It will be convenient to introduce the following notation:

<u>Definition</u>. $\bar{\mu} = \max(\mu_0, \mu_1)$

3. The Nonlinear Elliptic Problem.

Throughout the rest of the paper we will assume that hypotheses (I)-(IV) are satisfied. Let us consider

$$(L_k + \lambda)u_k - h^i_k u_i = f_k(x,U) \quad \text{in} \quad \Omega \tag{6}$$

$$B_k u_k - e^i_k u_i = g_k(x,U) \quad \text{on} \quad \Delta_k \tag{7}$$

$$u_k = \theta_k \quad \text{on} \quad \Gamma_k \tag{8}$$

Using the more concise notation we define the formal nonlinear operator N by defining $N(U) = V$ if V is a genralized solution of

$$(L + \lambda)V - HV = F(x,U) \quad \text{in} \quad \Omega ,$$

$$BV - EV = G(x,U) \quad \text{on} \quad \Delta ,$$

$$V = \Theta \quad \text{on} \quad \Gamma .$$

Then solving (6)-(8) is tantamount to finding a fixed point for N. We shall be interested in the case where F and G are dominated by affine functions. This is a reasonable assumption for many practical applications. For one thing it means that positive solutions to the associated parabolic equations (i.e. reaction-diffusion equations) grow no faster than exponentially, thus ensuring existence of a global solution whenever local solutions exist. In other words we want to assume that there exist a matrix $H_F(x)$ whose entries are all positive and some vector $D(x)$ such that

$$F(x,U) \leq H_F(x)U + D(x) . \tag{9}$$

Obviously, due to the presence of H on the left side of our equations we may as well subtract $H_F U$ on both sides, therefore assuming that F, and similarly G, are bounded from above for all U. As a specific example let us consider the case where one models the processes of chemical reactor kinetics or of flame propagation (see [4] for the equations). In both these cases one of the components is temperature and the boundary condition is obtained from heat flux consideration at the boundary. If a significant amount of heat is lost by radiation one expects a boundary condition of the form

$$K \frac{\partial u}{\partial \nu} = g(u) \equiv \alpha + \beta u - \gamma_1 (u^4 - u_0^4) \quad \text{on} \quad \partial\Omega \tag{10}$$

where $u \geq u_0$, u_0 being the temperature of the exterior region, $\alpha \geq 0$, and γ_1 is a positive constant obtained as the product of the emissivity of the container's surface and the Stefan-Boltzmann constant [3], and K is the heat conductivity. If on the other hand one assumes natural convection at the boundary one obtains [3]

$$K \frac{\partial u}{\partial \nu} = g(u) \equiv -\gamma_2 (u - u_0)^{5/4} , \tag{11}$$

where $\gamma_2 > 0$ and $u \geq u_0$. In the interval $0 < u < u_0$, the remaining physically meaningful range of the temperatures, one might have some other boundary condition which matches at u_0. In any case we notice that in both cases $g(u)$ is dominated by a linear function for $u \geq 0$. For $u < 0$ we can apparently define g to be whatever is convenient in order to satisfy mathematical hypotheses. That this causes no problems follows from a result which we shall prove which says that the existence theorem stated below is still valid even if the linear domination hypothesis fails in some region, provided some other condition holds. In the above example this condition amounts to observing that if we set $\tilde{u} \equiv 0$ we get

$$K \frac{\partial \tilde{u}}{\partial \nu} \leq g(\tilde{u}) .$$

When the corresponding partial differential equation is also nonhomogeneous we must require a similar inequality there. For example if we are dealing with a one-component case $Lu = f(u)$, we also require $L\tilde{u} \leq f(\tilde{u})$. Following standard terminology one may call $\tilde{u}$ a sub-solution, a term which we however have already used. In addition to domination by an affine map we also must require some reasonable local behavior.

Definition. Let (S,μ) be a measure space and T a function mapping $S \times R^m$, or a subset thereof, into R^k. Then T is said to satisfy the Caratheodory condition if $T(x,U)$ is measurable in x for each fixed $U \in R^m$ and is continuous in U for almost all x in S.

Definition. Let $\mathcal{S} \subset S \times R^m$ then $\mathcal{F}_r(\mathcal{S})$ denotes the class of all functions $T : \mathcal{S} \to R^m$ which satisfy the Caratheodory condition and also satisfy:

i) There exists a $D \in \mathcal{L}^r(S)$ such that $T(x,U) \leq D(x)$ for all $(x,U) \in \mathcal{S}$.

ii) For each real number ν there exists a $T_\nu \in L^r(S)$ such that $F(x,U) \geq T_\nu(x)$ for all $(x,U) \in S$ with $U \leq (\nu,\nu,\ldots,\nu)$

A simple example of a map $T \in F_r(\Omega \times R^m)$ is one which is continuous, nonincreasing and bounded from above. Another example is a continuous function which bounded. In particular if S is bounded and closed and T continuous on S then $T \in F_r(S)$ for for any $0 < r \leq \infty$. We introduce another hypothesis which will be needed for almost all subsequent results.

(V): There exist numbers $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, such that for all (x,U), (x,V)

$$(F(x,U) - F(x,V))\cdot(U - V) \leq \gamma_1|U - V|^2$$

$$(G(x,U) - G(x,V))\cdot(U - V) \leq \gamma_2|U - V|^2$$

Using the notation of lemma 2 (i) and (ii) we define $\gamma = \gamma_1 + \gamma_2 C\left(\frac{\varepsilon_0}{\gamma_2}\right)$.

<u>Theorem 9</u>. Suppose (I)-(IV) are satisfied, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$, $F \in F_{q/2}(\Omega \times R^m)$, $G \in F_p(\partial\Omega \times R^m)$ and $\lambda > \bar{\mu}$. Then (6)-(8) has a generalized solution. If (V) is also satisfied then this solution is unique for $\lambda > \mu_0 + \gamma$.

<u>Proof</u>: Let $D_F \in L^{q/2}(\Omega)$ and $D_G \in L^p(\partial\Omega)$ such that $F(x,U) \leq D_F(x)$ and $G(x,U) \leq D_G(x)$ for all $U \in R^m$. By Theorem 7 we know there exists a number $\nu > 0$ such that $N_\nu \equiv (\nu,\nu,\ldots,\nu) \geq \underset{\sim}{G}_\lambda(D_F,D_G,\Theta)$. Let

$$\mathfrak{K} = \{U \in H^1(\Omega) \,|\, \underset{\sim}{G}_\lambda(F_\nu,G_\nu,\Theta) \leq U \leq N_\nu\}\ ,$$

where $F(x,U) \geq F_\nu(x) \in L^{q/2}(\Omega)$ and $G(x,U) \geq G_\nu(x) \in L^p(\partial\Omega)$ for all $U \leq N_\nu$. Now $\mathfrak{K}$ is mapped into itself by N, for if $U \in \mathfrak{K}$ then

$$N(U) = \underset{\sim}{G}_\lambda(F(x,U),G(x,U),\Theta) \leq \underset{\sim}{G}_\lambda(D_F,D_G,\Theta) \leq N_\nu$$

and

$$N(U) \geq \underset{\sim}{G}_\lambda(F_\nu,G_\nu,\Theta)$$

It remains to prove that N is compact continuous (in the $H^1(\Omega)$-topology) because then the result follows from Schauder's fixed point theorem. Suppose $\{U_i\}$ is a sequence in $\mathfrak{K}$ which is bounded with respect to the norm $\|\ \|_1$ in $H^1(\Omega)$. We can, by Rellich's lemma, find a subsequence $\{U_{i'}\}$ which converges in $L^2(\Omega)$. Also, since $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is compact continuous, we may assume $\gamma_0 U_{i'}$ converges in

$L^2(\partial\Omega)$ (a fortiori to $\gamma_0 U$ where U is the $L^2(\Omega)$ limit of the sequence $\{U_{i'}\}$). We have (lemma 2)

$$\begin{aligned}
\varepsilon_0 \|N(U_{i'}) - N(U_{j'})\|_1^2 &\le A_\lambda(N(U_{i'}) - N(U_{j'}), N(U_{i'}) - N(U_{j'})) \\
&= \int_\Omega (F(x,U_{i'}) - F(x,U_{j'}))\cdot(N(U_{i'}) - N(U_{j'}))dx \\
&+ \int_{\partial\Omega} (G(x,U_{i'}) - G(x,U_{j'}))\cdot\gamma_0(N(U_{i'}) - N(U_{j'}))dS
\end{aligned}$$

By Theorem 7 $N(\mathfrak{K})$ is bounded in the norm $|||\ ||| \equiv \|\ \|_{1,2,\Omega} + \|\ \|_{0,\infty,\Omega}$ of $H \equiv H^1(\Omega) \cap L^\infty(\Omega)$. Therefore there exists a constant c such that

$$\varepsilon_0 \|N(U_{i'}) - N(U_{j'})\|_1^2 \le c\{\|F(x,U_{i'}) - F(x,U_{j'})\|_{0,1,\Omega} + \|G(x,U_{i'}) - G(x,U_{j'})\|_{0,1,\partial\Omega}\} \quad (12)$$

Since $\mathfrak{K}$ is a bounded set in $L^\infty(\Omega)$ the Nemytskii operator F takes $\mathfrak{K}$ into $L^{q/2}(\Omega)$. Similarly the image of $\gamma_0(\mathfrak{K})$ under the Nemytskii operator G is bounded in $L^p(\partial\Omega)$. But this means [14, p.22] that these operators, being defined through functions satisfying the Caratheodory condition, are continuous on $\mathfrak{K}$ and $\gamma_0(\mathfrak{K})$ in their respective L^1-topologies. Hence, by (12), $\{N(U_{i'})\}$ is a Cauchy sequence in $H^1(\Omega)$. We have incidentally shown that (12) also implies continuity. To prove uniqueness we suppose that $N(U) = U$ and $N(V) = V$, then using lemma 2 we get

$$\begin{aligned}
\varepsilon_0\|U-V\|_1^2 + (\lambda - \mu_0)\|U-V\|_0^2 &\le A_\lambda(U - V, U - V) \\
&= A_\lambda(N(U) - N(V), U - V) \le \gamma_1\|U-V\|_0^2 + \frac{1}{2}\varepsilon_0\|U-V\|_1^2 \\
&+ \gamma_2 C(\varepsilon_0/2\gamma_2)\|U-V\|_0^2 = \frac{1}{2}\varepsilon_0\|U-V\|_1^2 + \gamma\|U-V\|_0^2
\end{aligned}$$

Therefore, if $\lambda > \mu_0 + \gamma$ then $U = V$. ∎

Of course the above theorem is also valid if the conditions on F and G are replaced by $-F \in F_{q/2}(\Omega \times R^m)$ and $-G \in F_p(\partial\Omega \times R^m)$. The above result as well as the next theorem generalize similar results obtained in [12] for one-component equations.

Theorem 10. Suppose (I) - (IV) are satisfied, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$, F and G satisfy the Caratheodory condition on $\Omega \times R^m$ and $\partial\Omega \times R^m$ respectively, and $\lambda > \bar{\mu}$. Suppose there exist nonincreasing functions ϕ and ψ from R^1 into itself such that for all $k \geq k_0 > 0$

$$\limsup_{s \to \infty} k\phi(k\psi(s))/s < 1 \tag{13}$$

and for each $1 \leq i < m$ we have the growth conditions

$$f_i(x,u_1,u_2,\ldots,u_m), g_i(x,u_1,u_2,\ldots,u_m) \leq \phi(s) \quad \text{if} \quad u_j \geq s \ \forall j ,$$

$$f_i(x,u_1,u_2,\ldots,u_m), g_i(x,u_1,u_2,\ldots,u_m) \geq \psi(s) \quad \text{if} \quad u_j \leq s \ \forall j .$$

Then (6)-(8) has a generalized solution. If (V) is also satisfied and $\lambda > \mu_0 + \gamma$ then the solution is unique.

Remark. Simple examples of functions satisfying (13) are $\psi(s) = -a - b\max(0,s)^{\gamma_1}$ and $\phi(s) = a + b\max(0,-s)^{\gamma_2}$ where a,b,γ_1 and γ_2 are positive constants satisfying $\gamma_1\gamma_2 < 1$.

Proof: Let $N_1 = (1,1,\ldots,1) \in R^m$ and $k_1 = ||\underset{\sim}{G}_\lambda(N_1,N_1,|\Theta|)||_\infty + k_0$ and let

$$\mathfrak{K}_y = \{U \in H^1(\Omega) \mid k_1\psi(y) \leq u_i \leq y\}$$

where $y > 0$ is chosen so large that $k_1\phi(k_1(\psi(y))/y < 1$. Then N maps $\mathfrak{K}_y$ into itself. To see this we may assume without loss of generality that $\phi(0) = -\psi(0) \geq 1$.

$$N(U) = \underset{\sim}{G}_\lambda(F(x,U),G(x,U),\Theta) \geq \underset{\sim}{G}_\lambda(\psi(y)N_1,\psi(y)N_1,|\Theta|)$$

$$\geq \psi(y)\underset{\sim}{G}_\lambda(N_1,N_1,|\Theta|) \geq k_1\psi(y)N_1 .$$

Also

$$N(U) \leq \underset{\sim}{G}_\lambda(\phi(k_1\psi(y))N_1,\phi(k_1\psi(y)N_1,|\Theta|) \leq \phi(k_1\psi(y))k_1N_1 \leq yN_1.$$

As in the proof of theorem 9 we have all the necessary components to justify the use of Schauders fixed point theorem. Uniqueness follows from the same argument that was used in the proof of the previous theorem. ∎

We conclude this section with a theorem on invariant sets which constitutes the crucial ingredient in the proof of the invariant set theorem for the reaction-diffusion equations discussed in the next section. Instead of viewing the result as an invariant set theorem one might, maybe more appropriately so, regard it is a nonlinear generalizational lemma 5, i.e. as a sort of maximum principle. With this in mind one would expect to need the following conditions: i) the i^{th} component of $F(x,U) + HU$ is nondecreasing in u_j for each $j \neq i$ (corresponding to the hypothesis $H \geq 0$ in lemma 5) and ii) (V) is satisfied (corresponding to the coerciveness requirement of A_λ. Also in order to be able to treat nonlinearities of the type occurring in (10) and (11) we certainly want to allow $f_i(x,U)$ to decrease "rapidly" with respect to u_i. This last requirement has tended to make our proof rather lengthy. Before we proceed we must introduce some more notation.

<u>Definitions</u>. (i) $\hat{F}(x,U) = F(x,U) + H(x)U$, $\hat{F} = (\hat{f}_1,\hat{f}_2,\ldots,\hat{f}_m)$

$\hat{G}(x,U) = G(x,U) + E(x)U$, $\hat{G} = (\hat{g}_1,\hat{g}_2,\ldots,\hat{g}_m)$

(ii) We use $+\infty$ (resp. $-\infty$) to also denote the extended real valued function $x \to +\infty$ (resp. $x \to -\infty$). For convenience we define $(L_i + \lambda)(\pm\infty) = \pm\infty$ and $B_i(\pm\infty) = \pm\infty$.

(iii) $\Phi = (\varphi_1,\varphi_2,\ldots,\varphi_m)$ where $\varphi_i \in H^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$ for $1 \leq i \leq d$ and $\varphi_i = -\infty$ for $i > d$. $\Psi = (\psi_1,\psi_2,\ldots,\psi_m)$ where $\psi_i \in H^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$ for $\delta \leq i \leq \ell$ and $\psi_i = +\infty$ for $i < \delta$ or $i > \ell$. Also we assume the indexing is such that $\delta \leq d + 1$. In other words the indices $1 \leq i < \delta$ are those for which φ_i is finite valued and ψ_i is $+\infty$, the indices $\delta \leq i \leq d$ are those for which both φ_i and ψ_i are finite valued, the indices $d + 1 \leq i \leq \ell$ are those for which φ_i is $-\infty$ but ψ_i is finite valued and the indices $\ell < i \leq m$ are those for which both $\varphi_i = -\infty$ and $\psi_i = +\infty$. We also use $[\Phi,\Psi]$ to denote $\{U \in H^1(\Omega) \mid \Phi \leq U \leq \Psi\}$.

(iv) $$\mathcal{S}_\Phi^\Psi = \{(x,U) \in \Omega \times R^m \mid \Phi(x) \leq U \leq \Psi(x)\}$$

$$\partial\mathcal{S}_\Phi^\Psi = \{(x,U) \in \partial\Omega \times R^m \mid \gamma_0\Phi(x) \leq U \leq \gamma_0\Psi(x)\}$$

(v) For any $U \in H^1(\Omega)$, $U_\Phi = (\varphi_1,\varphi_2,\ldots,\varphi_d,u_{d+1},\ldots,u_m)$ and $U^\Psi = (u_1,u_2,\ldots,u_{\delta-1},\psi_\delta,\psi_{\delta+1},\ldots,\psi_\ell,u_{\ell+1},\ldots,u_m)$. That is to say U_Φ is obtained from Φ by replacing all components which are $-\infty$ by corresponding components from U and similarly U^Ψ is obtained from Ψ by replacing components which are $+\infty$ by corresponding components from U.

<u>Theorem 11</u>. Suppose that (I)-(V) are satisfied and $\lambda > \bar{\mu} + \gamma$, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$, $F \in \mathcal{F}_{q/2}(S_\Phi^\Psi)$, $G \in \mathcal{F}_p(\partial S_\Phi^\Psi)$ and that $\hat{f}_i(x,u_1,u_2,\ldots,u_m)$ and $\hat{g}_i(x,u_1,u_2,\ldots,u_m)$ are nondecreasing in u_j for all $1 \leq j \leq \ell$ with $j \neq i$. Suppose $\Phi \leq \Theta \leq \Psi$ and that for all $U \in [\Phi,\Psi]$:

$$\text{(IQ)} \quad \begin{aligned} (L+\lambda)\Phi &\leq HU_\Phi + F(x,U_\Phi) \text{ and } (L+\lambda)\Psi \geq HU^\Psi + F(x,U^\Psi) \text{ in } \Omega , \\ B\Phi &\leq EU_\Phi + G(x,U_\Phi) \text{ and } B\Psi \geq EU^\Psi + G(x,U^\Psi) \text{ on } \Delta . \end{aligned}$$

Then (6)-(8) has a unique generalized solution $\bar{U} \in [\Phi,\Psi]$

We shall postpone the proof until the end of this section. This theorem can be viewed as an invariant set theorem in the following way. For λ sufficiently large let $T_\lambda : L^{q/2}(\Omega) \times L^p(\partial\Omega) \times H \to H \cap C(\Omega)$ be the operator defined by $T_\lambda(F_0,G_0,\Theta_0) = V$ where V is the unique solution of

$$\begin{aligned} (L+\lambda-H)V - F(x,V) &= F_0 \quad \text{in } \Omega , \\ (B-E)V - G(x,V) &= G_0 \quad \text{on } \Delta , \\ V &= \Theta_0 \quad \text{on } \Gamma . \end{aligned}$$

The fact that $V \in C(\Omega)$ follows from known regularity results [15, p. 201]. Suppose $F \in \mathcal{F}_{q/2}(\Omega \times R^m)$, $G \in \mathcal{F}_p(\partial\Omega \times R^m)$ and that the inequalities (IQ) are satisfied for all $U \in [\Phi,\Psi]$. Then for fixed $\Theta_0 \in [\Phi,\Psi]$, and $\mu > 0$ the map $W \to T_{\lambda+\mu}(\mu W, 0, \Theta_0)$ leaves $[\Phi,\Psi]$ invariant. The proof of this follows immediately from the inequalities $(L+\lambda+\mu)\Phi \leq HU_\Phi + F(x,U_\Phi) + \mu W$ and $(L+\lambda+\mu)\Psi \geq HU^\Psi + F(x,U^\Psi) + \mu W$. It is also easy to prove the following generalization of lemma 5.

<u>Corollary 12</u>. Suppose (I)-(V) are satisfied, $\lambda > \bar{\mu} + \gamma$, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$, $F \in \mathcal{F}_{q/2}(\Omega \times R^m)$, $G \in \mathcal{F}_p(\partial\Omega \times R^m)$ and that for all $1 \leq i \leq \ell$: $\hat{f}_i(x,u_1,u_2,\ldots,u_m)$ and $\hat{g}_i(x,u_1,u_2,\ldots,u_m)$ are nondecreasing in u_j for all $j \neq i$. Then T_λ is an order preserving map i.e. if

$$F_1 \geq F_0, \quad G_1 \geq G_0 \quad \text{and} \quad \Theta_1 \geq \Theta_0 \quad \text{then } T_\lambda(F_1,G_1,\Theta_1) \geq T_\lambda(F_0,G_0,\Theta_0) .$$

To prove this let $\Phi = T_\lambda(F_0,G_0,\Theta_0)$ and $U = T_\lambda(F_1,G_1,\Theta_1)$ and apply the theorem.

In Corollary 12 we have a lot of monotonicity available. At the other extreme we may delete the monotonicity requirement entirely from the statement of Theorem 11 provided we replace (IQ) the requirement that for all $U \in [\Phi,\Psi]$

$$L_k\varphi_k + \lambda\varphi_k \le h_k^i u_i + f_k(x,u_1,u_2,\ldots,u_{k-1},\varphi_k,u_{k+1},\ldots,u_m) \quad (1 \le k \le d) ,$$

$$B_k\varphi_k \le e_k^i u_i + g_k(x,u_1,u_2,\ldots,u_{k-1},\varphi_k,u_{k+1},\ldots,u_m) \quad (1 \le k \le d) ,$$

$$L_k\psi_k + \lambda\psi_k \ge h_k^i u_i + f_k(x,u_1,\ldots,u_{k-1},\psi_k,u_{k+1},\ldots,u_m) \quad (\delta \le k \le \ell) ,$$

$$B_k\psi_k \ge e_k^i u_i + g_k(x,u_1,\ldots,u_{k-1},\psi_k,u_{k+1},\ldots,u_m) \quad (\delta \le k \le \ell) ,$$

yielding a result akin to Theorem 8 in [13]. Since we only assume that the inequalities are satisfied for $U \in [\Phi,\Psi]$, instead of for all $U \in H^1(\Omega)$, this result is not just a repeated application of the theorem. We will return to this point with a remark at the end of this section.

The following lemma will be necessary for the proof of Theorem 11.

Lemma 13. Suppose $u \in H^1_S(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$ and that $G = \{x \in \Omega | u(x) > 0\}$ and $R = S \cap \partial G$. Then the restriction of u to G is a member of $H^1_R(G)$.

Proof: Let

$$E_k = \{x \in \Omega | 1/k < |u(x)| \le 2/k\} .$$

then there must exist a subsequence $\{k(n)\}$ of positive integers such that $\lim_{n\to\infty} m(E_{k(n)}) = 0$, where m is the usual Lebesque measure on Ω. We define

$$\xi_n(x) = \max[0,\min(1,2 - k(n)|u(x)|] ,$$

a function which is a member of $H^1(\Omega)$ and is equal to 0 whenever $|u(x)| \ge 2/k(n)$ and equal to 1 when $|u(x)| \le 1/k(n)$. Moreover

$$D_i\xi_n(x) = \begin{cases} -\mathrm{sgn}(u(x))k(n)D_iu(x) & \text{if } x \in E_{k(n)} \\ 0 & \text{if } x \in \Omega\backslash E_{k(n)} \end{cases}$$

One easily verifies that $D_i\xi_n u = uD_i\xi_n + \xi_n D_i u$. We first show that $\xi_n u \to 0$ in $H^1(\Omega)$. Let

$$S_n = \{x \in \Omega | |u(x)| \le 2/k(n)\}$$

then we have

$$\int_\Omega (\xi_n u)^2 dx \le \int_{S_n} (\xi_n u)^2 dx \le 4m(\Omega)/k(n)^2 ,$$

while

$$\int_\Omega (D_i\xi_n u)^2 dx \le 2\int_\Omega [(D_i\xi_n)^2u^2 + (D_iu)^2\xi_n^2]dx \le$$

$$2\int_{E_{k(n)}} k(n)^2(D_iu)^2u^2dx + 2\int_{S_n}(D_iu)^2dx \le$$

$$8\int_{E_{k(n)}} (D_iu)^2dx + 2\int_{S_n}(D_iu)^2dx$$

We note that since $m(\Omega) < \infty$ the last term tends to $2\int_{S_\infty}(D_iu)^2dx$ where $S_\infty = u^{-1}(\{0\})$. But by the remarks made just before the proof of lemma 4 it follows that this integral is zero. The next to last term tends to zero because $m(E_{k(n)}) \to 0$. There exists a function $J : (0,\infty) \to (0,\infty)$ such that

$$\max_{1\le i\le n}\int_\sigma (D_iu)^2dx < \varepsilon \quad \text{whenever} \quad m(\sigma) < J(\varepsilon) .$$

Since $u \in H^1_S(\Omega)$ there exists a sequence $\{u_n\} \subset H^1(\Omega)$ such that for each positive integer n, there exists an open neighborhood N_n of $\partial\Omega\backslash S$ such that u_n vanishes on N_n. We may assume without loss of generality that there exists a positive number K such that $|u_n(x) - u(x)| \le K$ a.e. for $n = 1,2,\ldots,$ and that $|u(x) - u_n(x)| \le 1/k(n)$ except on a set σ_n of measure less than $J(1/k(n)^3)$. Clearly $(1-\xi_n)u_n$, restricted to G, is a member of $H^1(G)$ which vanishes on a neighborhood of $\partial G\backslash S$. We observe that

$$u - (1-\xi_n)u_n = -\xi_n(u-u_n) + (u-u_n) + \xi_n u ,$$

where the last two terms tend to zero in $H^1(\Omega)$. Obviously $\xi_n(u-u_n)$ tends to zero in the $L^2(\Omega)$ topology so that we only need to examine convergence of its derivatives.

$$\int_\Omega \{D_i[\xi_n(u-u_n)]\}^2dx \le \int_{E_{k(n)}} 2(D_i\xi_n)^2(u-u_n)^2dx + \int_\Omega 2\xi_n^2[D_i(u-u_n)]^2dx \le$$

$$\int_{E_{k(n)}\cap(\Omega\backslash\sigma_n)} 2(D_iu)^2dx + \int_{E_{k(n)}\cap\sigma_n} 2k(n)^2|D_iu|^2K^2dx + 2\|u-u_n\|_1^2$$

$$\le \int_{E_{k(n)}} 2(D_iu)^2dx + 2K^2/k(n) + 2\|u-u_n\|_1^2 \to 0 \quad \text{as} \quad n\to\infty . \quad \blacksquare$$

Remark. In the proof of Theorem 11 we will use the following fact. Suppose we alter E and H by setting certain entries equal to zero. This changes the quadratic functional $A_\lambda(U,U)$ to a new one, $A^*_\lambda(U,U)$. Let $U^* = (|u_1|,|u_2|,\ldots,|u_m|)$, then

$$A^*_\lambda(U,U) \geq A^*_\lambda(U^*,U^*) \geq A_\lambda(U^*,U^*)$$

$$\geq \varepsilon_0\|U^*\|_1^2 - \mu_0\|U^*\|_0^2 = \varepsilon_0\|U\|_1^2 - \mu_0\|U\|_1^2$$

Hence all the results which we have proven are still true, for the same values of λ, for the problem obtained by setting one or more of entries h_k^i and e_k^i equal to zero.

Proof of Theorem 11: We will use $u \vee v$ to denote the function $x \to \max(u(x),v(x))$ and if $U = (u_1,u_2,\ldots,u_m)$ and $V = (v_1,v_2,\ldots,v_m)$ then $U \vee V = (u_1 \vee v_1, u_2 \vee v_2,\ldots,u_m \vee v_m)$. We similarly define the greatest lower bounds $u \wedge v$ and $U \wedge V$. Next we would like to introduce a notation which can be used to denote certain matrices obtained from H and E by replacing one or more columns by columns of zeros. If $M = (m_{ij})$ is an $m \times m$ matrix then the matrix ${}^{[k,r]}M = ({}^{[k,r]}m_{ij})$ is the matrix defined by

$${}^{[k,r]}m_{ij} = \begin{cases} m_{ij} & \text{if } k \leq j \leq r\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $S \geq 0$ be a member of $H^1(\Omega) \cap L^\infty(\Omega)$ which we shall choose later. Next we define $F_0 : \Omega \times R^m \to R^m$ and $G_0 : \partial\Omega \times R^m \to R^m$ by

$$F_0(x,U) = F(x,(U \vee \Phi) \wedge \Psi) + {}^{[1,\delta-1]}H(x)((U \vee \Phi) \wedge S) +$$
$${}^{[\delta,d]}H(x)((U \vee \Phi) \wedge \Psi) + {}^{[d+1,\ell]}H(x)((U \wedge \Psi) - U)\,,$$

and similarly

$$G_0(x,U) = G(x,(U \vee \Phi) \wedge \Psi) + {}^{[1,\delta-1]}E(x)((U \vee \Phi) \wedge S) +$$
$${}^{[\delta,d]}E(x)((U \vee \Phi) \wedge \Psi) + {}^{[d+1,\ell]}E(x)((U \wedge \Psi) - U)\,.$$

Let $\varphi^* = \max\limits_{1\leq i\leq d} \|\varphi_i\|_{0,\infty,\Omega}$ and $\psi^* = \max\limits_{\delta\leq i\leq \ell} \|\psi_i\|_{0,\infty,\Omega}$. We note that the first term appearing on the right hand side of the definition of F_0 is a member of $F_{q/2}(\Omega \times R^m)$. The second term is $\leq HS$ and, for $U \leq (\nu,\nu,\ldots,\nu)$, where ν is a real number, it is bounded from below by $-H(x)(\varphi^*,\varphi^*,\ldots,\varphi^*) \in L^{q/2}(\Omega)$. Hence the second term is a member of $F_{q/2}(\Omega \times R^m)$. The third term also belongs to $F_{q/2}(\Omega \times R^m)$ because it is bounded from above by $H(x)(\psi^*,\psi^*,\ldots,\psi^*) \in L^{q/2}(\Omega)$ and from below by $-H(x)(\varphi^*,\varphi^*,\ldots,\varphi^*) \in L^{q/2}(\Omega)$. Finally the last term is ≤ 0 and, for $U \leq (\nu,\nu,\ldots,\nu)$ is bounded from below by $H((0,0,\ldots,0) \wedge (-\nu - \psi^*,-\nu - \psi^*,\ldots,-\nu - \psi^*))$. Hence $F_0 \in F_{q/2}(\Omega \times R^m)$. Similarly we have $G_0 \in F_p(\partial\Omega \times R^m)$. Therefore we may apply Theorem 9, which says that we have a (generalized) solution U_0 to the problem

$$\begin{aligned} (L + \lambda)U_0 - {}^{[d+1,m]}HU_0 &= F_0(x,U_0) && \text{in } \Omega \\ BU_0 - {}^{[d+1,m]}EU_0 &= G_0(x,U_0) && \text{on } \Delta \\ U_0 &= \Theta && \text{on } \Gamma \end{aligned}$$

Let $0 \leq \overline{F} \in L^{q/2}(\Omega)$ be an upper bound for the sum of the first, third and fourth terms in the definition of F_0. Similarly $0 \leq \overline{G} \in L^p(\partial\Omega)$ is an upper bound for the sum of the first, third and fourth terms in the definition of G_0. Let S be the solution of

$$\begin{aligned} (L + \lambda)S - HS &= \overline{F} + {}^{[\delta,d]}HS && \text{in } \Omega \\ BS - ES &= \overline{G} + {}^{[\delta,d]}ES && \text{on } \Delta \\ S &= \Theta \vee 0 && \text{on } \Gamma \end{aligned}$$

Applying lemma 5 to

$$\begin{aligned} (L + \lambda)(S - U_0) - {}^{[d+1,m]}H(S - U_0) &\geq 0 && \text{in } \Omega \\ B(S - U_0) - {}^{[d+1,m]}E(S - U_0) &\geq 0 && \text{on } \Delta \\ S - U_0 &\geq 0 && \text{on } \Gamma \end{aligned}$$

we get $S \geq U_0$. But this means that

$$\begin{aligned} (L + \lambda)U_0 &\leq \hat{F}(x,(U_0 \vee \Phi) \wedge \Psi) && \text{in } \Omega \\ BU_0 &\leq \hat{G}(x,(U_0 \vee \Phi) \wedge \Psi) && \text{on } \Delta \\ U_0 &= \Theta && \text{on } \Gamma \end{aligned}$$

Let $\delta_i = \psi_i - u_{0i}$. For each $1 \leq i \leq m$

$$\begin{aligned} (L_i + \lambda)\psi_i &\geq \hat{f}_i(x,(U_0 \vee \Phi)^{\Psi}) && \text{in } \Omega \\ B\psi_i &\geq \hat{g}_i(x,(U_0 \vee \Phi)^{\Psi}) && \text{on } \Delta_i \\ \psi_i &\geq \theta_i && \text{on } \Gamma_i \end{aligned}$$

Let $G_k = \{x | u_{0k}(x) > \psi_k(x)\}$. We claim that $G_k = \phi$. Suppose this were not the situation. We can show $\delta_k \wedge 0$ is a member of $H^1_{\Delta_k}(\Omega)$. To see this we first observe that since $u_{0k} - \theta_k \in H^1_{\Delta_k}(\Omega)$ there exists a sequence $\{v_j\} \subset H^1(\Omega)$ such that $v_j \to u_{0k} - \theta_k$ in $H^1(\Omega)$ and such that $v_j = 0$ on a neighborhood of Γ_k. But then $[\psi_k - (v_j + \theta_k)] \wedge 0 \to \delta_k \wedge 0$ in $H^1(\Omega)$ as $j \to \infty$, which implies that $\delta_k \wedge 0 \in H^1_{\Delta_k}(\Omega)$. Applying the lemma $\delta_k \in H^1_{R_k}(G)$ where $R_k = \Delta_k \cap \partial G$. Therefore,

$$(L_k + \lambda)\delta_k \geq \hat{f}_k(x, u_{01} \vee \varphi_1, \dots, u_{0\delta-1} \vee \varphi_{\delta-1}, \psi_\delta, \dots, \psi_\ell, u_{0\ell+1}, \dots, u_{0m})$$

$$- \hat{f}_k(x, u_{01} \vee \varphi_1 \wedge \psi_1, \dots, u_{0\ell} \vee \varphi_\ell \wedge \psi_\ell, u_{0\ell+1}, \dots, u_{0m}) \geq 0 \quad \text{in } G_k$$

$$B_k\delta_k \geq \hat{g}_k(x, u_{01} \vee \varphi_1, \dots, u_{0\delta-1} \vee \varphi_{\delta-1}, \psi_\delta, \dots, \psi_\ell, u_{0\ell+1}, \dots, u_{0m})$$

$$- \hat{g}_k(x, u_{01} \vee \varphi_1 \wedge \psi_1, \dots, u_{0\ell} \vee \varphi_\ell \wedge \psi_\ell, u_{0\ell+1}, \dots, u_{0m}) \geq 0 \quad \text{on } R_k$$

$$\delta_k = 0 \quad \text{on } \partial G \setminus R_k ,$$

where we used the monotonicity and the fact that $(u_{0k} \vee \varphi_k) \wedge \psi_k = \psi_k$ on G. Actually some care must be taken to verify that the boundary condition on R_k is truly satisfied for the problem on G. To prove this we first show that if $u \in H^1_{R_k}(G)$ then $\bar{u} \in H^1_{R_k}(\Omega)$ where we define $\bar{u}$ simply as

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}$$

In order to do this we may, without loss of generality, assume that $u(x) = 0$ on a neighborhood N of $\partial G \setminus R_k$. Let $v \in \mathcal{D}(\Omega)$, and define $\bar{v}$ to be a function in $\mathcal{D}(\Omega)$ which agrees with v on $\operatorname{supp} u \cap \operatorname{supp} v$ and such that $\operatorname{supp} \bar{v} \subset G$. This is possible since $\operatorname{supp} u \cap \operatorname{supp} v$ and ∂G disjoint compact sets. Now

$$\int_{\Omega G} \bar{u} D_i v dx = \int_G u D_i \bar{v} dx = -\int_G (D_i u)\bar{v} dx$$

Hence, for each i, $D_i\bar{u}(x)$ equals $D_i u(x)$ on G and is 0 outside G, i.e. $D_i\bar{u} \in L^2(\Omega)$. Moreover $\bar{u} = 0$ on a neighborhood of $\partial\Omega \setminus R_k$, namely $\bar{\Omega}\setminus \operatorname{supp} u$. Now, since δ_k is defined on all of Ω and satisfies $B_k\delta_k = B_k\psi_k - B_k u_{0_k}$ on Δ_k, (in the generalized sense via A_λ) it follows that the boundary condition is satisfied on R_k i.e. via the bilinear functional A_λ defined on $H^1(G) \times H^1(G)$. Therefore, by lemma 5, $\delta_k \geq 0$ on G_k which implies $G_k = \phi$. Hence $U_0 \leq \Psi$. Applying the inequalities which we know hold for $U_{0\Phi}$ we obtain

$$(L + \lambda)(U_0 - U_{0\Phi}) \geq \hat{F}(x, U_0 \vee \Phi) - \hat{F}(x, U_{0\Phi}) \quad \text{in } \Omega\,,$$
$$B(U_0 - U_{0\Phi}) \geq \hat{G}(x, U_0 \vee \Phi) - \hat{G}(x, U_{0\Phi}) \quad \text{on } \Delta\,,$$
$$(U_0 - U_{0\Phi}) \geq 0 \quad \text{on } \Gamma\,.$$

Using an argument entirely analogous to the one used to show that $U_0 \leq \Psi$, we obtain from the above inequalities the fact that $U_0 \geq \Phi$, thus concluding the proof of the theorem since U_0 also solves (6)-(8).

<u>Remark</u>. Suppose one has several pairs $(\Phi^{(j)}, \Psi^{(j)})$, $1 \leq j \leq r$, as in the statement of Theorem 11, and suppose that

$$(L + \lambda)\Phi^{(j)} \leq \hat{F}(x, U_{\Phi^{(j)}}) \quad \text{and} \quad (L + \lambda)\Psi^{(j)} \geq \hat{F}(x, U^{\Psi^{(j)}}) \quad \text{in } \Omega\,,$$
$$B\Phi^{(j)} \leq \hat{G}(x, U_{\Phi^{(j)}}) \quad \text{and} \quad B\Psi^{(j)} \geq \hat{G}(x, U^{\Psi^{(j)}}) \quad \text{on } \Delta\,,$$

and that $\Phi^{(j)} \leq \Theta \leq \Psi^{(j)}$ for all $1 \leq j \leq m$ and all $U \in [\Phi, \Psi]$ where $\Phi = \Phi^{(1)} \vee \Phi^{(2)} \vee \ldots \vee \Phi^{(r)}$ and $\Psi = \Psi^{(1)} \wedge \Psi^{(2)} \wedge \ldots \wedge \Psi^{(r)}$. Then there exists a solution $U_0 \in [\Phi, \Psi]$ to (6)-(8). To see this one merely notes that the first part of the proof of Theorem 11 still shows there exists a solution U_0 to

$$(L + \lambda)U_0 = \hat{F}(x, (U_0 \vee \Phi) \wedge \Psi) \quad \text{in } \Omega$$
$$BU_0 = \hat{G}(x, (U_0 \vee \Phi) \wedge \Psi) \quad \text{on } \Delta$$
$$U_0 = \Theta \quad \text{on } \Gamma$$

Next we note that

$$(L_i + \lambda)\psi_k^{(j)} \geq \hat{f}_k(x, ((U_0 \vee \Phi) \wedge \Psi)^{\Psi^{(j)}}) \quad \text{in } \Omega$$

and a corresponding inequality on Δ. Letting $\delta_k = \psi_k^{(j)} - u_{0k}$ we obtain the appropriate inequalities for δ_k which show that $u_{0k} \leq \psi_k^{(j)}$. Hence $U_0 \leq \Psi$ and similar arguments lead to the conclusion that $U_0 \geq \Phi$. ∎

4. The Nonlinear Parabolic Problem.

We turn our attention to the system

$$\frac{\partial u_k}{\partial t} + L_k u_k = \hat{f}_k(x,t,U) \quad \text{in } \Omega \times (0,T) , \tag{14}$$

$$B_k u_k = \hat{g}_k(x,U) \quad \text{on } \Delta_k \times (0,T) , \tag{15}$$

$$u_k(x,t) = \theta_k(x) \quad \text{on } \Gamma_k \times (0,T) , \tag{16}$$

$$u_k(x,0) = u_k^0(x) \quad \text{in } \Omega . \tag{17}$$

We assume that the only explicit time dependence appears in the $\hat{f}_k$'s, although this can be generalized. For example, if the coefficients of L_k are regular enough, then we can allow time dependence in the principal coefficients without complicating matters too much. Time dependent boundary conditions however seem to lead to more serious difficulties.

In order to obtain our results we shall employ the nonlinear semigroup theory of Crandall, Liggett, and Pazy [7], [8]. This seems to be appropriate for the investigation of invariant sets since this type of semigroup "lives" on a closed set which does not necessarily have to be an entire Banach space. We first briefly describe the nonlinear semigroup results which will be used.

Let X be a Banach space and for each $t \geq 0$ let $A(t)$ be an operator from $D(t) \subset X$, its domain, into X which satisfies

$$\|x + \lambda A(t)x - (y + \lambda A(t)y)\| \geq (1 - \lambda\omega)\|x - y\|$$

for all $x,y \in D(t)$ and all $0 < \lambda < 1/\omega$, where ω is some given positive number. Suppose that the closure, $\overline{D(t)}$, of the domain is independent of time and

$$\overline{D(t)} = \overline{D(0)} \subset \text{Range } (I + \lambda A(t)) \quad \text{for all } 0 \leq t < T ,$$

and all $0 < \lambda < 1/\omega$. Finally we suppose that $J_\lambda(t) \equiv (I + \lambda A(t))^{-1}$ satisfies

$$\|J_\lambda(t)x - J_\lambda(\tau)x\| \leq \lambda|\mu(t) - \mu(\tau)|M(\|x\|)$$

for all $0 \leq t , \tau < T$ and $x \in \overline{D(t)}$, where $\mu : [0,T] \to X$ is a continuous function of bounded variation and $M : [0,\infty) \to [0,\infty)$ is a non-decreasing function. Under these assumptions

$$\mathcal{U}(t,s)x \equiv \prod_{i=1}^{n} J_{(t-s)/n}(s + i(t-s)/n)x$$

exists for all $x \in \overline{\mathcal{D}(0)}$, $0 \leq s < t < T$ and

$$\lim_{t \downarrow s} \mathcal{U}(t,s)x = x \qquad \forall\, x \in \overline{\mathcal{D}(0)}\ .$$

$\mathcal{U}(t,s)$ is called the propagation operator because if $y : [0,T) \to X$ is a continuous, strongly differentiable map satisfying

$$\frac{dy}{dt} + \mathcal{A}(t)y = 0 \qquad y(s) = y_0 \in \overline{\mathcal{D}(0)}$$

then $y(t) = \mathcal{U}(t,s)y_0$ [8, Theorem 3.1].

Our aim will be to find an invariant set which is equal to $\overline{\mathcal{D}(0)}$ for an appropriate nonlinear semigroup. This means that we must find $\mathcal{D}(t)$ such that (15) is satisfied. This makes it necessary to exactly determine the domain of the operator L. The difficulty in this lies in the interpretation of the boundary condition (2). Since $a_k^{ij} \in L^\infty(\Omega)$ and $D_i u_k \in L^2(\Omega)$ their traces on $\partial\Omega$ are not well defined. However we can circumvent this problem as follows. Define $\tilde{B}$ to be the unique linear operator

$$\tilde{B} : \mathcal{V}(L) \equiv \{U \in H^1(\Omega) \,|\, LU - HU \in L^2(\Omega)\} \to H^{-1/2}(\Omega)$$

which satisfies

$$A_0(U,V) - (LU - HU,V) = (\tilde{B}U,\gamma_0 V)_{\partial\Omega}$$

for all $V \in H^1(\Omega)$. The existence of $\tilde{B}$ is easily established via the Riesz representation theorem [2], and one can also check to see that if U and the coefficients of L,H,B, and E are sufficiently well behaved then

$$\tilde{B}_k u_k = \nu_i [a_k^{ij} D_j u_k + d_k^i u_k] + \sigma_k u_k - e_k^i u_i$$

where the right hand side can be evaluated pointwise. Since $H^1(\Omega)$ is a Hilbert space there exists an orthogonal projection operator $\pi_\Delta : H^1(\Omega) \to H^1(\Omega)$ with $\pi_\Delta H^1(\Omega) = H^1_\Delta(\Omega)$. Suppose $\alpha \in H^{1/2}(\partial\Omega)$. Let $\mathrm{U} \in \gamma_0^{-1}(\alpha)$ and define $\tilde{\pi}_\Delta \alpha = \gamma_0 \pi_\Delta \mathrm{U} \in H^{1/2}_\Delta(\partial\Omega) \equiv \gamma_0 H^1_\Delta(\Omega) \subset H^{1/2}(\partial\Omega)$. This is a well defined map since if $\gamma_0 U = \gamma_0 U'$ then $U - U' \in H^1_0(\Omega) \subset H^1_\Delta(\Omega)$ so that $\gamma_0 \pi_\Delta(U - U') = \gamma_0(U - U') = 0$. Hence we have a projection operator $\tilde{\pi}_\Delta$ satisfying $\tilde{\pi}_\Delta \gamma_0 = \gamma_0 \pi_\Delta$. We also have the corresponding adjoints $\pi^*_\Delta : H^1(\Omega)' \to H^1(\Omega)'$ and $\tilde{\pi}^*_\Delta : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

Lemma 14. Suppose $F \in L^2(\Omega)$, $G \in H^{-1/2}(\partial\Omega)$ and $\Theta \in H^1(\Omega)$, then U is a generalized solution of (1)-(3) iff.

(a) $(L + \lambda)U - HU = F$ (as distributions)

(b) $\tilde{\pi}^*_\Delta[\tilde{B}U - G] = 0$

(c) $\pi_\Delta[U - \Theta] = U - \Theta$

Proof: If U is a generalized solution then

$$A_\lambda(U,V) = (F,V) + (G,V)_{\partial\Omega} \quad \forall V \in H^1_\Delta(\Omega)$$

In particular

$$A_\lambda(U,V) = (LU + \lambda U - HU,V) = (F,V)$$

for all $V \in C^\infty(\Omega)$ with compact support in Ω, and therefore (a) is satisfied. By the definition of $\tilde{B}$

$$A_\lambda(U,V) = (LU + \lambda U - HU,V) + (\tilde{B}U,\gamma_0 V)_{\partial\Omega} = (F,V) + (G,\gamma_0 V)_{\partial\Omega}$$

for all $V \in H^1_\Delta(\Omega)$, and hence

$$(\tilde{B}U,\gamma_0\pi_\Delta V)_{\partial\Omega} = (G,\gamma_0\pi_\Delta V) \quad \forall V \in H^1(\Omega)$$

so that

$$\gamma_0^*\tilde{\pi}_\Delta^*[\tilde{B}U - G] = \pi_\Delta^*\gamma_0^*[\tilde{B}U - G] = 0$$

Since γ_0 is surjective, hence γ_0^* injective, (b) follows. Because $U - \Theta \in H^1_\Delta(\Omega)$, $\pi_\Delta(U - \Theta) = U - \Theta$. Conversely suppose (a), (b), (c) are satisfied. Obviously $U - \Theta \in H^1_\Delta(\Omega)$. Using the definition of $\tilde{B}$ together with (a), (b) and the fact that $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$ yields

$$A_\lambda(U,V) = (F,V) + (G,\gamma_0 V)_{\partial\Omega} \quad \forall V \in H^1_\Delta(\Omega)\,. \qquad \blacksquare$$

Returning to the problem (14)-(17) we see that the above lemma implies that

$$A(t) : U \to LU - \hat{F}(x,t,U)$$

is a well defined operator from

$$\{U \in H^1(\Omega)\,|\,LU \in L^2(\Omega), \pi_\Delta(U - \Theta) = U - \Theta, \hat{F}(x,t,U) \in L^2(\Omega)\,,$$
$$\tilde{\pi}^*_\Delta[\tilde{B}U - j\hat{G}(x,U)] = 0\}$$

into $L^2(\Omega)$ where $j : L^p(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$. The closure of this set will, if the coefficients of L and B are "nice", be all of $L^2(\Omega)$. In order to obtain invariant sets of the type described in the introduction we will instead define the domain by

$$\mathcal{D} = U \in H^1(\Omega) \,|\, LU \in L^2(\Omega), \pi_\Delta(U-\Theta) = U-\Theta, \tilde{\pi}^*_\Delta[\tilde{B}U - j\hat{G}(x,U)] = 0, \hat{F}(x,t,U) \in L^2(\Omega), \Phi \leq U \leq \Psi\} .$$

The following hypothesis will also be needed.

(VI): (i) There exists a constant $K > 0$ such that

$$|\hat{F}(x,t,U) - \hat{F}(x,\tau,U)| \leq K|U||t-\tau|$$

We also assume F satisfies (V) with γ_1 independent of t, and $\hat{F} \in F_2(S^{\Psi}_{\Phi})$.

(ii) There exists a collection $\mathcal{D}_{00} \subset (\mathcal{D}(\Omega))^m$ such that $\mathcal{D}_{00}$ is a dense subset (with respect to the $L^2(\Omega)$-topology) of

$$\mathcal{D}_0 \equiv \{U \in H^1(\Omega) \,|\, LU - HU \in L^2(\Omega), \pi_\Delta U = U, \tilde{\pi}^*\tilde{B}U = 0\}$$

Condition (i) is more restrictive than needed. Condition (ii) is a technical necessity which can be replaced by additional regularity requirements on the coefficients. For example, if a^{ij}_k and d^i_k are of class $C^1(\bar{\Omega})$ then we can set $\mathcal{D}_{00} = (\mathcal{D}(\Omega))^m$. If these coefficients are sectionally C^1 with discontinuities across surfaces in Ω whose union Γ has a closure whose n-dimensional measure is zero then $\mathcal{D}_{00}$ can be taken to be the collection of all $C^\infty(\Omega)$ functions with compact support in $\Omega\backslash\bar{\Gamma}$.

Please recall that $\hat{F}(x,t,U) = H(x,t)U + F(x,t,U)$ and $\hat{G}(x,U) = E(x)U + G(x,U)$.

Lemma 15. Suppose (I)-(VI) are satisfied, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$, and for each fixed $t \in [0,T)$, with $T < \infty$, we have $F \in F_{q/2}(S^{\Psi}_{\Phi})$ and $G \in F_p(\partial S^{\Psi}_{\Phi})$. For $1 \leq i \leq \ell$ we suppose $\hat{f}_i(x,t,u_1,u_2,\ldots,u_m)$ and $\hat{g}_i(x,u_1,u_2,\ldots,u_m)$ are nondecreasing in u_j for all $j \neq i$, $1 \leq j \leq \ell$. Finally suppose that $\Phi \leq \Theta \leq \Psi$ and that for all $U \in [\Phi,\Psi]$

$$L\Phi \leq \hat{F}(x,t,U_\Phi) \quad \text{and} \quad L\Psi \geq \hat{F}(x,t,U^\Psi) \quad \text{in } \Omega \times [0,T)$$

$$B\Phi \leq \hat{G}(x,U_\Phi) \quad \text{and} \quad B\Psi \geq \hat{G}(x,U^\Psi) \quad \text{on } \Delta \times [0,T)$$

Then $A(t)$ satisfies:

(i) $$\|(U + \lambda A(t)U) - (V + \lambda A(t)V)\|_0 \geq (1-\lambda\omega)\|U - V\|_0$$

for all $U,V \in \mathcal{D}$ and all $0 < \lambda < \omega^{-1}$, where ω is some fixed positive number.

(ii) The $L^2(\Omega)$-closure of $\mathcal{D}$ is $\bar{\mathcal{D}} = \{U \in L^2(\Omega) \,|\, \Phi \leq U \leq \Psi\}$, a set we will denote by $\overline{[\Phi,\Psi]}$.

(iii) Range $(I + \lambda \mathcal{A}(t)) \supset \overline{[\Phi,\Psi]}$

(iv) If $W(t) + \lambda \mathcal{A}(t)W(t) = F_0 \in \overline{[\Phi,\Psi]}$ (i.e. $W(t) = J_\lambda(t)F_0$) then

$$\|W(t_1) - W(t_2)\|_0 \leq C|t_1 - t_2|(\|F\|_0 + 1)$$

for all $t_1, t_2 \in [0,T)$, where C is a constant.

<u>Proof:</u> (i) $\|U - V\|_0 \|U + \lambda \mathcal{A}(t)U - V - \lambda \mathcal{A}(t)V\|_0 \geq$

$$(U - V, U - V) + \lambda A_0(U - V, U - V) - \lambda(F(x,t,U) - F(x,t,V), U - V)$$

$$- \lambda(G(x,U) - G(x,V), U - V)_{\partial\Omega} \geq [1 - \lambda(\mu_0 + \gamma m(\Omega))]\|U - V\|_0^2 ,$$

where we used (VI), lemma 2, and the definition of γ.

(ii) Let U be the generalized solution of

$$(L + \lambda)U = \hat{F}(x,t,U) \quad \text{in } \Omega , \quad (\lambda > \bar{\mu}) ,$$

$$BU = \hat{G}(x,U) \quad \text{on } \Delta ,$$

$$U = \Theta \quad \text{on } \Gamma .$$

Suppose W is any element in $\mathcal{D}_{00}$ such that $\Phi \leq U + W \leq \Psi$. Since $W \equiv 0$ on some neighborhood of $\partial\Omega$ we have

(a) $(L + \lambda - H)(U + W) = F(x,t,U) + (L + \lambda - H)W \in L^2(\Omega), \hat{F}(x,t,U + W) \in L^2(\Omega)$,

(b) $\tilde{\pi}^*_\Delta[\tilde{B}(U + W) - j\hat{G}(x,\gamma_0(U + W))] =$

$\tilde{\pi}^*_\Delta[\tilde{B}U - j\hat{G}(x,\gamma_0 U)] = 0$,

(c) $\pi_\Delta(U + W - \Theta) = W + \pi_\Delta(U - \Theta) = W + U - \Theta$.

Hence $\mathcal{D} \supset \{U + W | W \in \mathcal{D}_{00}, \Phi \leq U + W \leq \Psi\}$, which upon taking closure with respect to the $L^2(\Omega)$ topology yields

$$\bar{\mathcal{D}} \supset \{U + W | W \in L^2(\Omega), \Phi \leq U + W \leq \Psi\} = \overline{[\Phi,\Psi]}$$

(iii) Let $F_0 \in \overline{[\Phi,\Psi]}$ and consider

$$\frac{1}{\lambda} U + LU = \hat{F}(x,t,U) + \frac{1}{\lambda} F_0 \quad \text{in } \Omega ,$$

$$BU = \hat{G}(x,U) \quad \text{on } \Delta ,$$

$$U = \Theta \quad \text{on } \Gamma .$$

By Theorem 11 we have a generalized solution $U \in [\Phi,\Psi]$ if $1 - (\bar{\mu} + \gamma)\lambda > 0$.

(iv) Let $W_i + \lambda A(t_i)W_i = F_i$ $(i = 1,2)$. Then

$$(\tfrac{1}{\lambda} - \mu_0)\|W_1 + W_2\|_0^2 + \varepsilon_0\|W_1 - W_2\|_1^2 \le A_{1/\lambda}(W_1 - W_2, W_1 - W_2)$$

$$\le (F(x,t_1,W_1) - F(x,t_2,W_1), W_1 - W_2) + (F(x,t_2,W_1) - F(x,t_2,W_2), W_1 - W_2)$$

$$+ (G(x,W_1) - G(x,W_2), W_1 - W_2)_\Delta + \tfrac{1}{\lambda}(F_1 - F_2, W_1 - W_2)$$

$$\le C'\{|t_1 - t_2|\,\|W_1\|_0\|W_1 - W_2\|_0 + \|W_1 - W_2\|_0^2 + \|W_1 - W_2\|_{0,2,\partial\Omega}^2\}$$

$$+ \tfrac{1}{\lambda}\|F_1 - F_2\|_0\|W_1 - W_2\|_0 .$$

Using lemma 2 we see that there exists a constant $C'(\varepsilon_0)$ such that the above inequality implies

$$(\tfrac{1}{\lambda} - \mu_0 - C'(\varepsilon_0))\|W_1 - W_2\|_0 + \varepsilon_0/2\|W_1 - W_2\|_1 \le C'|t_1 - t_2|\,\|W_1\|_0 + \tfrac{1}{\lambda}\|F_1 - F_2\|_0 \qquad (18)$$

First we let $t_2 = t_0$, some fixed value in $(0,T)$, and $W_2 = W_0$, the solution corresponding to the case where $F_2 = 0$. We then obtain

$$(\tfrac{1}{\lambda} - \mu_0 - C'(\varepsilon_0))\|W_1\| \le (\tfrac{1}{\lambda} - \mu_0 - C'(\varepsilon_0))\|W_0\| + C'T\|W_1\|_0 + \tfrac{1}{\lambda}\|F_1\|$$

or

$$\|W_1\| \le \frac{(1 - \lambda[\mu_0 + C'(\varepsilon_0)])\|W_0\| + \|F_1\|}{1 - \lambda_0[\mu_0 + C'(\varepsilon_0) + C'T]} \qquad (\lambda < \lambda_0)$$

where λ_0 is chosen so small that the denominator is larger than $\frac{1}{2}$. Hence we have

$$\|W_1\| \le 2\|W_0\| + 2\|F_1\|$$

Returning to inequality (18) and selling $F_1 = F_2 = F$ obtain, for $0 < \lambda < \lambda_0$,

$$\|J_\lambda(t_1)F - J_\lambda(t_2)F\| = \|W_1 - W_2\| \le 4\lambda C'|t_1 - t_2|\{\|W_0\| + \|F\|\}.$$

This concludes the proof of the lemma which guarantees the existence of a propagation operator for a nonlinear semigroup on $\overline{[\Phi,\Psi]}$ generated by A. ∎

Definitions. $U : [0,T) \to L^2(\Omega)$ is called a strong solution of (14)-(17) if

(a) U is continuous on $[0,T)$ and $U(0) = U^0$.

(b) U is absolutely continuous on compact subsets of $(0,T)$.

(c) U is differentiable almost everywhere on $(0,T)$ and is a generalized solution of (14)-(16) (regarded as an elliptic problem) for almost all $t \in (0,T)$.

A subset $\mathcal{S} \subset L^2(\Omega)$ is called an invariant set for (14-(16) if $U(t) \in \mathcal{S}$ for all $t \in (s,T)$ whenever U is a strong solution of (14)-(17) with $U(s) \in \mathcal{S}$.

<u>Theorem 16</u>. Suppose the hypotheses of lemma 15 are satisfied (except $T = \infty$ is allowed). Then there exists a propagation operator $\mathcal{U}(t,s)$ defined on $[\overline{\Phi,\Psi}]$ corresponding to problem (14)-(16). In particular $[\overline{\Phi,\Psi}]$ is an invariant set for this problem. Moreover, if the graph of $A(t)$ is closed then $\mathcal{U}(t,0)U^0$ is a strong solution for each $U^0 \in D$.

<u>Proof</u>. The existence of $\mathcal{U}(t,s)$ follows from the lemma. By Theorem 3.1 in [8] any strong solution U of (14)-(17) with $U(s) \in [\overline{\Phi,\Psi}]$ must satisfy $U(t) = \mathcal{U}(t,s)U^0$, $t \geq s$, and hence $[\overline{\Phi,\Psi}]$ is an invariant set. The last assertion of the theorem follows from Theorem 3.4 in [8]. ■

As in the elliptic case there are various possible corollaries we could state. One such result was stated in the introduction. We shall state two more.

<u>Corollary 17</u>. Suppose (I)-(VI) are satisfied, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$, Φ and Ψ are members of $H^1(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, F and G are continuous on $\overline{\Omega} \times R^{m+1}$ and $\overline{\Omega} \times R^m$ respectively and for each $1 \leq k \leq m$ $\hat{f}_k(x,t,u_1,u_2,\ldots,u_m)$ and $\hat{g}_k(x,u_1,u_2,\ldots,u_m)$ are nondecreasing on u_j for $j \neq k$. Suppose $\Phi \leq \Theta \leq \Psi$ and

$$L\Phi \leq \hat{F}(x,t,\Phi) \quad \text{and} \quad L\Psi \geq \hat{F}(x,t,\Psi) \quad \text{in } \Omega \times [0,T)$$

$$B\Phi \leq \hat{G}(x,\Phi) \quad \text{and} \quad B\Psi \geq \hat{G}(x,\Psi) \quad \text{on } \Delta \times [0,T)$$

Then $[\overline{\Phi,\Psi}]$ is an invariant set and $\mathcal{U}(t,0)U^0$ is a strong solution whenever $U^0 \in D \subset [\Phi,\Psi]$.

<u>Proof</u>: We only have to establish that $A(t)$ is closed. Suppose $U_n \in D$ and $U_n \to U$ in $L^2(\Omega)$ and $A(t)U_n \equiv F_n \to F$ in $L^2(\Omega)$. Using (18) one easily sees that this means $U_n \to U$ in $H^1(\Omega)$ and hence $\gamma_0 U_n \to \gamma_0 U$ in $L^2(\partial\Omega)$. But since F and G satisfy the Caratheodory condition for each t this means $F(x,t,U_n) \to F(x,t,U)$ in $L^{q/2}(\Omega)$ while $G(x,U_n) \to G(x,U)$ in $L^p(\partial\Omega)$. To see this we use the fact that the U_n's are uniformly bounded and a standard continuity result for Nemytskii operators [14, p. 22]. Using (18) once again we see that $J_\lambda(t)$, and hence $A(t)$, is closed. ■

We also have the following result for the case where we have no monotonicity requirement on the coupling

Corollary 18. Suppose (I)-(VI) are satisfied, $\Theta \in H^1(\Omega) \cap L^\infty(\Omega)$ and for each fixed $t \in [0,T)$ we have $F \in F_{q/2}(S_\Phi^\Psi)$ and $G \in F_p(\partial S_\Phi^\Psi)$ where $\Phi \leq \Theta \leq \Psi$. Then $\overline{[\Phi,\Psi]}$ is an invariant set provided that for all $U \in [\Phi,\Psi]$ we have

$$L_k\varphi_k \leq \hat{f}_k(x,u_1,u_2,\ldots,u_{k-1},\varphi_k,u_{k+1},\ldots,u_m) \quad (1 \leq k \leq d) ,$$

$$B_k\varphi_k \leq \hat{g}_k(x,u_1,u_2,\ldots,u_{k-1},\varphi_k,u_{k+1},\ldots,u_m) \quad (1 \leq k \leq d) ,$$

$$L_k\psi_k \geq \hat{f}_k(x,u_1,u_2,\ldots,u_{k-1},\psi_k,u_{k+1},\ldots,u_m) \quad (\delta \leq k \leq \ell) ,$$

$$B_k\psi_k \geq \hat{g}_k(x,u_1,u_2,\ldots,u_{k-1},\psi_k,u_{k+1},\ldots,u_m) \quad (\delta \leq k \leq \ell) .$$

These inequalities are essentially requirements that the "velocity" on the "faces" $\{U|u_k = \varphi_k\}$ and $\{U|u_k = \psi_k\}$ are in the right direction. If one has monotonicity this "velocity" only needs to be checked at the "edges" $\{U|u_i = \varphi_i,\ 1 \leq i \leq d\}$ and $\{U|u_i = \psi_i,\ \delta \leq i \leq \ell\}$ (the statement of the theorem) while in the extreme case of totally monotonic coupling (Corollary 17) we only need to check the "velocities" at the "vertices" Φ and Ψ.

Proof: By the remark at the end of the section on elliptic systems we see that it suffices for the inequalities to hold for all $U \in [\Phi,\Psi]$. Hence part (iii) of lemma 15 is still true. The other parts of lemma 15 are obviously also still true since the relevant hypotheses are those which this lemma and Theorem 11 have in common. Therefore the proof of Theorem 16 again applies here. ■

In conclusion we mention that these results can be generalized to problems involving even more general, but still time independent, boundary conditions on Lipschitz continuous boundaries. We can also allow time dependence in the elliptic operators L_k provided the coefficients are sufficiently regular. This is done by applying the full power of the semigroup results in [8].

APPENDIX

Theorem. Suppose that $u \in H^1(\Omega)$ and that for all v in $H^1(\Omega)$

$$a_\lambda(u,v) \equiv \int_\Omega [a^{ij}(D_j u)(D_i v) + d^i u D_i v + b^i v D_i u + (c+\lambda)uv]dx = \int_{\partial\Omega} gv\, dS .$$

where we assume that (I)-(III) are satisfied. (Since we are dealing with the one component case, $m = 1$, the subscript k is deleted), that $\lambda > \tilde{\mu}$ and $g \in L^p(\partial\Omega)$. Then $u \in L^\infty(\Omega)$.

Lemma. ([19, p. 24]). Let $\zeta = \zeta(t)$ be a nonnegative, nonincreasing function on the half line $t \geq 0$ such that there are positive constants C, α and β such that

$$\zeta(h) \leq C(h-k)^{-\alpha}[\zeta(k)]^\beta \quad \text{for } h > k \geq 0 .$$

Then, if $\beta > 1$, there exists a constant $d \geq 0$ such that $\zeta(d) = 0$ (e.g. $d = C^{1/\alpha}[\zeta(0)]^{(\beta-1)/\alpha}2^{\beta(\beta-1)}$).

Proof of the Theorem: Let $v = (\operatorname{sgn} u)\max(|u| - k, 0) = (u-k) \vee 0 + (u+k) \wedge 0 \in H^1(\Omega)$. We have, letting $E(k) = \{x \in \bar{\Omega} \,|\, |u(x)| \geq k\}$:

$$a_\lambda(u,v) = \Big(\int_{E(k)} + \int_{\Omega\setminus E(k)}\Big)\big((a^{ij}D_j u + d^i u)D_i v + (b^i D_i u + (c+\lambda)u)v\big)dx$$

$$= \int_{E(k)} \big((a^{ij}D_j v + d^i v)D_i v + v b^i D_i v + (c+\lambda)v^2\big)dx$$

$$+ k \int_{\{u(x)\geq k\}} (d^i D_i v + cv + \lambda v)dx - k \int_{\{u(x)\leq -k\}} (d^i D_i v + cv + \lambda v)dx$$

$$= a_\lambda(v,v) + k\int_{E(k)} (d^i D_i |v| + c|v| + \lambda|v|)dx$$

$$= a_\lambda(v,v) + k\int_\Omega (d^i D_i |v| + c|v| + \lambda|v|)dx$$

$$= a_\lambda(v,v) + k\int_\Omega (c + \lambda - D_i d^i)|v| + k\int_{\partial\Omega} \nu_i d^i |v| dS$$

Hence $a_\lambda(u,v) \geq a_\lambda(v,v)$. Also there exists a constant $K \geq 0$ such that

$$\|v\|_1^2 \leq K a_\lambda(v,v) \leq K a_\lambda(u,v) = K\int_{\partial\Omega} gv\, dS$$

If we set $F(k) = E(k) \cap \partial\Omega$ then

$$\int_{\partial\Omega} gv\, dS = \int_{F(k)} gv\, dS \leq \|g\|_{0,r,F(k)} \|\gamma_0 v\|_{0,\rho,F(k)} \leq C_0 \|g\|_{0,r,F(k)} \|v\|_1 ,$$

where $\rho = 2(n-1)/(n-2)$ and $r = 2(n-1)/n$. We note that $p > r$, hence another application of Hölders inequality yields

$$\|g\|_{0,r,F(k)} \leq \|g\|_{0,p,F(k)}\, m(F(k))^{1/r-1/p} .$$

Using the Sobolev inequality $\|\gamma_0 v\|_{0,\rho,\partial\Omega} \leq C_0\|v\|_1$ and the fact that $v = 0$ on $\partial\Omega\backslash F(k)$ we have

$$\|\gamma_0 v\|_{0,\rho,F(k)} \leq KC_0^2\|g\|_{0,p,\partial\Omega} m(F(k))^{1/r-1/p}$$

Hence if $h > k$ then

$$m(F(h))^{1/\rho}(h-k) \leq \|\gamma_0 v\|_{0,\rho,F(k)} \leq KC_0^2\|g\|_{0,p,\partial\Omega}\, m(F(k))^{1/r-1/p}$$

Letting $\zeta(h) = m(F(h))$ we have, for $h > k > 0$

$$\zeta(h) \leq (KC_0^2\|g\|_{0,p,\partial\Omega})^{\rho}(h-k)^{-\rho}\zeta(k)^{(1/r-1/p)\rho}$$

An application of the lemma concludes the proof.

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In this paper we consider systems of weakly coupled nonlinear second order elliptic and parabolic equations with nonlinear, possibly coupled, boundary conditions. The aim is to find invariant sets of the form

$$S = \{(u_1,u_2,\ldots,u_m) \mid \varphi_i(x) \le u_i(x) \le \psi_i(x) \quad \text{a.e.}\}$$

for certain nonlinear reaction-diffusion equations:

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$$U_t + LU = F(U) \quad \text{in} \quad \Omega ,$$
$$BU = G(U) \quad \text{on} \quad \partial\Omega ,$$

where $L = (L_1, L_2, \ldots, L_m)$ (L_i a linear second order elliptic operator) and $B = (B_1, B_2, \ldots, B_m)$ (B_i a linear boundary operator of a general type) and $U = (u_1, u_2, \ldots, u_m)$. The main result essentially says that $S = \{U \mid \Phi \leq U \leq \Psi\}$ is an invariant set if

$$L\Phi \leq F(\Phi) \quad \text{and} \quad L\Psi \geq F(\Psi) \quad \text{in} \quad \Omega$$

and

$$B\Phi \leq G(\Phi) \quad \text{and} \quad B\Psi \geq G(\Psi) \quad \text{on} \quad \partial\Omega .$$

The work also includes some existence results for the parabolic problem and the associated nonlinear elliptic problem.